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Term Structure of Interest Rates, Yield Curve Residuals, and the Consistent Pricing of Interest Rates and Interest Rate Derivatives*

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Term Structure of Interest Rates, Yield Curve Residuals, and the Consistent Pricing of Interest Rates and Interest Rate Derivatives

ABSTRACT

Dynamic term structure models (DTSMs) price interest rate derivatives based on the model-implied fair values of the yield curve, ignoring any pricing residuals on the yield curve that are either from model approximations or market imperfections. In contrast, option pricing in practice often takes the market observed yield curve as given and focuses exclusively on the specification of the volatility structure of forward rates. Thus, if any errors exist on the observed yield curve, they will be carried over permanently. This paper proposes a new framework that consistently prices both interest rates and interest rate derivatives. In particular, under such a framework, instead of making *a priori* assumptions, we allow the data on interest rates and interest rate derivatives to dictate the dynamics of the yield curve residuals, as well as their impact on the pricing of interest rate derivatives.

Specifically, we propose an $m+n$ model structure. The first m factors capture the systematic movement of the yield curve and hence are referred to as the *yield curve factors*. The latter n factors are derived from the residuals on the yield curve and are labeled as the *residual factors*. We estimate a simple $3+3$ Gaussian affine example using eight years of data on U.S. dollar LIBOR/swap rates and interest rate caps. The model performs well in pricing both interest rates and interest rate derivatives. Furthermore, we find that small residuals on the yield curve can have large impacts on the pricing of interest rate caps. Under the estimated model, the three Gaussian yield curve factors explain over 99.5 percent of the variation on the yield curve, but only account for less than 25 percent of the variation in the cap implied volatility. Incorporating the three residual factors improves the explained variance in cap implied volatility to over 95 percent. We investigate the reasons behind the "amplification" of yield curve residuals in pricing interest rate derivatives and find that the yield curve residuals are a recurring phenomenon, not a one-time event. Hence, the dynamics of the residuals influence option prices even if the current residual level is zero. We also find that the residuals concentrate on the two ends of the yield curve and are more transient than the original interest rate series, both of which, we argue, contribute to the amplification effect.

JEL CLASSIFICATION CODES: E43, G12, G13, C51.

KEY WORDS: term structure; yield curve; interest rate caps; implied volatility; residual factors; extended Kalman Filter; quasi-maximum likelihood estimation.

Term Structure of Interest Rates, Yield Curve Residuals, and the Consistent Pricing of Interest Rates and Interest Rate Derivatives

In pricing interest rates and interest rate derivatives, the literature has been taking two distinct approaches . The first approach, often referred to as dynamic term structure models (DTSMs), captures the dynamics of the yield curve with a finite-dimensional state vector. Empirical studies have found that a well-designed dynamic term structure model can explain over 90 percent of the variation on the yield curve with as few as three factors. Refer to Dai and Singleton (2002b) for a comprehensive review on the theory, estimation, and performance of DTSMs. These models, however, have shown limited success in pricing interest rate derivatives.[1] A distinct feature of these models is that they price interest rate derivatives based on the model-implied fair values of the yield curve, thus ignoring any potential impacts on the future payoff calculation from the residuals on the yield curve, which occur naturally in a finite dimensional framework, either due to model approximations or market imperfections. In reality, however, the terminal payoffs of options are settled based on market observed rates, not on model implied fair values.

In practice, the task of pricing interest rate derivatives has mostly been handled by an alternative approach, which takes the yield curve as given and focuses exclusively on the pricing of interest rate derivatives.[2] Yet, by taking the yield curve as given, these models focus exclusively on the pricing of interest rate derivatives, and hence have little to say about the fair valuation and time series dynamics of the underlying interest rates. Furthermore, accommodating the whole yield curve often necessitates accepting an *infinite dimensional* state space and/or time-inhomogeneous model parametrization, both of which create difficulties for hedging practices. Besides, the "over-fitting" of the yield curve implies that, if any errors exist on the observed yield curve, they will be carried over permanently.

This paper proposes an $m+n$ model structure that bridges the gap between the two existing approaches by successfully pricing both interest rates and interest rate derivatives within a finite dimen-

[1] Recent endeavors in pricing interest rate derivatives based on DTSMs include Singleton and Umantsev (2001) and Jagannathan, Kaplin, and Sun (2001).

[2] Examples include the forward rate models of Ho and Lee (1986), Hull and White (1993), and Heath, Jarrow, and Morton (1992), market rate models of Brace, Gatarek, and Musiela (1997), Glasserman and Kou (2000), Jamshidian (1997), Miltersen, Sandmann, and Sondermann (1997), and Musiela and Rutkowski (1997a), and string models of Goldstein (2000), Santa-Clara and Sornette (2001), Longstaff, Santa-Clara, and Schwartz (2001b), and Han (2001).

sional framework. Under this framework, the first m factors capture the systematic movement of the yield curve and hence are referred to as the *yield curve factors*. The latter n factors are derived from the residuals on the yield curve and are labeled as the *residual factors*.

The key innovation of the $m+n$ model structure lies in the direct modeling of the dynamics of the yield curve residuals and the impact of the residual dynamics on the pricing of interest rate derivatives. Under a low dimensional term structure model, residuals are bound to be observed on the yield curve, either due to model approximations or market imperfections, such as bid-ask spread, non-synchronous trading, misquotes, or other temporary market imperfections that have not been arbitraged away. Dynamic term structure models compute future payoffs based on the model-implied fair values, as if residuals are non-existent or negligible; the option pricing frameworks take the entire initial yield curve as given, hence forcing the observed yield curve to the "fair value" and carrying any potential mispricing on the yield curve permanently into the future. In contrast, our $m+n$ model neither ignores the residuals as immaterial, nor does the model assumes them permanent. Instead, we allow the time series of interest rates and interest rate derivatives to dictate the dynamics of these residuals. In this regard, our model is more flexible than both practices in the literature.

We elaborate on the model structure through a simple $3+3$ independent Gaussian affine example. We first use three independent Gaussian affine factors to price the yield curve, and then model the residuals on the yield curve and their impacts on cap pricing with another three independent Gaussian factors. We estimate the model using eight years (April 1994 to April 2002) of data on U.S. dollar LIBOR/swap rates and cap implied volatilities. The estimation is performed using quasi maximum likelihood method jointly with extended Kalman filter via a two stage procedure. In the first stage, we extract the three yield curve factors and estimate the dynamics of the yield curve factors using the LIBOR and swap rates. In the second stage, we extract the residual factors and estimate their dynamics based on the measurement errors on the yield curve obtained from the first stage and the market quotes on cap implied volatilities.

Despite its simple structure, the model performs well in pricing both interest rates and interest rate derivatives. The three yield curve factors explain over 99.5 percent of the variation on the yield curve. By incorporating three yield curve residual factors, the model also explains over 95 percent of the aggregate variation in cap implied volatilities. Furthermore, the estimation results indicate that,

although residuals on the yield curve are very small, their impact on option pricing is quite large. The residuals account for less than one percent of the variation on the yield curve, but the three residual factors contribute to over 70 percent of the variation in cap implied volatility. Without the three residual factors, the three yield curve factors only explain less than 25 percent of the variation in cap implied volatility.

Further analysis of the yield curve residuals reveals several features that contribute to their large impacts on option pricing. First, we find that the yield curve residuals are a recurring phenomenon, rather than a one-time event. If we regard the model-implied fair value as an equilibrium state, the residual estimates indicate that this equilibrium is not an absorbing state. Thus, the dynamics of the residuals and their impacts on option pricing warrant careful consideration. Second, while residuals on moderate maturity swap rates are very small, the residuals on very short and very long maturity interest rates are significantly larger. The two ends of the yield curve dictate the curvature of the yield curve, which, in turn, has been found to influence the option implied volatility movements the most (See Heidari and Wu (2001)). Third, the yield curve residuals are much more transient than the yield curve factors and hence have a much larger impact on the *conditional* dynamics of interest rates than on the unconditional dynamics. Since option prices reflect the *conditional* dynamics of interest rates, the impacts of the more transient yield curve residuals are significantly larger on options than on the yield curve.

The fact that yield curve factors alone cannot account for the movement of interest rate options has been documented and interpreted differently in the literature. In a joint statistical analysis on LIBOR/swap rates and cap/swaption implied volatilities, Heidari and Wu (2001) find that three principal components extracted from the yield curve explain over 99 percent of the interest rate movements, but only explain 60 percent of the variation on the swaption/cap implied volatility surface. They further find that three additional principal components extracted from the interest rate options are needed to explain the "independent" movement in the implied volatility surface. Collin-Dufresne and Goldstein (2002) document similar evidence and refer to it as "unspanned stochastic volatility (USV)." To explain the USV, they identify a set of parameter constraints within the affine family of term structural models so that the stochastic volatility of interest rates is not instantaneously correlated with the value of the interest rate. Singleton and Umantsev (2002) propose an alternative explanation based on the nature of

volatility risk premium, also within the affine family. They argue that there are shocks to the volatility of yields under the risk-neutral measure that do not affect the correlation structure of swaps under the objective measure. Fan, Gupta, and Ritchken (2002) investigate the issue from the perspective of hedging. They find that if one is allowed to recalibrate a term structure model daily so as to match the current yield curve and volatility surface, there will not be much implied volatility movement left unspanned by the yield curve. This evidence tentatively suggests that the implied volatility exhibit movements independent only of the smoothed yield curve factors, but not of the market observed yield curve. Hence, residuals on the yield curve may play important roles in the pricing and hedging of interest rate derivatives. Our $m+n$ model explicitly accommodates the role played by the yield curve residuals. As a result, a simple $3+3$ independent Gaussian affine example can successfully price both interest rates and interest rate derivatives.

The paper is structured as follows. The next section elaborates on the model structure through a $3+3$ independent Gaussian affine example. Section II addresses the data issue and the estimation procedure. Section III discusses the performance of the $3+3$ Gaussian affine example. Section IV investigates the dynamics of the yield curve residuals and its impacts on pricing interest rate derivatives. Section V concludes with discussions on future research.

I. The $m+n$ Model

We argue that residuals on the yield curve can have important impacts on the pricing of interest rate derivatives. For this purpose, we propose an $m+n$ model structure, where the first m factors capture the systematic movement of the yield curve while the additional n factors control the dynamics of the residuals on the yield curve. We first fix the notation and then elaborate on the model structure via a simple, concrete example.

A. Set-up and Notation

We fix a filtered complete probability space $\{\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{0\leq t\leq \mathcal{T}}\}$ satisfying the usual technical conditions with $\mathcal{T}$ being some finite, fixed time. Let F and E denote two vector Markov processes in some

state space $\mathcal{D}_m \in \mathbb{R}^m$ and $\mathcal{D}_n \in \mathbb{R}^n$, respectively. We assume that the observed market quotes on interest rates, y_t, are subject to the following heuristic decomposition,

$$y_t = Y(F_t) + \varepsilon(E_t),$$

where $Y(F_t)$ denotes the "fair value" of the interest rate, the dynamics of which is controlled by the state vector F_t, and $\varepsilon(E_t)$ denotes the residuals of the interest rate, which we assume to be governed also by a finite-dimensional state vector E_t. We christen F_t as the **yield curve factors** and E_t the **residual factors**. We assume that the two state vectors are independent of each other and satisfy the following stochastic differential equations under measure $\mathbb{P}$:

$$dF_t \quad = \quad \mu(F_t)dt + \Sigma(F_t)dW_t, \quad dE_t = \mu(E_t)dt + \Sigma(E_t)dZ_t, \tag{1}$$

where $\mu(F_t)$ is an $m \times 1$ vector defining the drift and $\Sigma(F_t)$ is an $m \times m$ matrix defining the diffusion of the F process. Similarly, $\mu(E_t)$ is an $n \times 1$ vector and $\Sigma(E_t)$ is an $n \times n$ matrix defining the drift and diffusion of the E process. W_t and Z_t are independent Brownian motions with dimension m and n, respectively.

For any time $t \in [0, \mathcal{T}]$ and time-of-maturity $T \in [t, \mathcal{T}]$, let $P(F_t, T)$ denote the "fair value" at time t of a zero-coupon bond with maturity $\tau = T - t$. Note that the fair value of the bond is assumed to be only a function of yield curve factors, but independent of the residual factors. The fair values of the spot rates are defined as

$$Y(F_t, T) \equiv \frac{1}{T-t} \ln P(F_t, T),$$

and the fair value of the instantaneous interest rate, or the short rate, r, is defined by continuity:

$$r(F_t) \equiv \lim_{T \downarrow t} \frac{-\ln P(F_t, T)}{T-t}.$$

We assume that there exists a measure, $\mathbb{P}^*$, under which the time-t fair value of a claim to a terminal payoff Π_T at time $T > t$ can be written as

$$V(F_t, E_t, T) = \mathbb{E}_t^* \left[\exp\left(-\int_t^T r(F_s) ds \right) \Pi_T \right], \tag{2}$$

where $\mathbb{E}_t^*[\cdot]$ denotes expectation conditional on filtration $\mathcal{F}_t$ and under measure $\mathbb{P}^*$. Thus, the fair value of a zero coupon bond can be computed from (2) by setting $\Pi_T = 1$ for all states. In particular, since the payoff of a zero bond is a constant and hence state independent, the fair value of the zero bond is only a function of the fair value of the short rates during the life of the bond and thus independent of the residual factors E, consistent with the original assumption.

Nevertheless, for state-contingent claims such as caps, the payoff at time T is determined by the market *observed* interest rates at that time (or one period earlier if paid in arrears). Thus, the payoff function Π_T will depend upon both the dynamics of the systematic yield curve factors F and that of the residual factors E. As a result, the value of the state contingent claim will become a function of the residual factors as well, in addition to its dependence on the yield curve factors. In what follows, we elaborate the pricing and estimation of our $m+n$ model structure through a simple, concrete model.

B. A $3+3$ Gaussian Affine Example

To illustrate the idea of our modeling structure, we construct and estimate a simple model within the analytically attractive Gaussian affine family. Specifically, we assume that both the yield curve factor F and the residual factor E have dimensions of three: $m = n = 3$. A yield curve dimension of three is consistent with the empirical evidence of Litterman and Scheinkman (1991), Knez, Litterman, and Scheinkman (1994), Longstaff, Santa-Clara, and Schwartz (2001a), and Heidari and Wu (2001). The choice of three residual factors is mainly motivated by the evidence in Heidari and Wu (2001) and Wadhwa (1999) on swaption implied volatilities.

We assume that, under the objective measure $\mathbb{P}$, the yield curve factors and the residual factors are both governed by Ornstein-Uhlenbeck (OU) processes,

$$dF_t = -\kappa_F F_t dt + dW_t, \quad dE_t = -\kappa_E E_t dt + dZ_t \tag{3}$$

where $\kappa_F \in \mathbb{R}^{3\times 3}$ and $\kappa_E \in \mathbb{R}^{3\times 3}$ controls the mean reversions of the two vector processes. For identification reasons, we normalize both state vectors to have zero long run means and identity diffusion matrix. We further assume that the fair value of the short rate r is affine in the yield curve factor,

$$r(F_t) = a_r + b_r^\top F_t, \tag{4}$$

where the parameters $a_r \in \mathbb{R}$ is a scalar and $b_r \in \mathbb{R}^{3+}$ is a vector.[3] Finally, we close the model by assuming an affine market price of risk

$$\gamma(F_t) = b_\gamma + \kappa_\gamma F_t \tag{5}$$

with $b_\gamma \in \mathbb{R}^3$ and $\kappa_\gamma \in \mathbb{R}^{3\times 3}$. We assume that the residual factors are not priced. Our specification is analogous to a three-factor Vasicek (1977) model except that we allow time varying market risk premium. The affine market price of risk specification is also applied in recent works by Duffee (2002), Dai and Singleton (2002a), and Liu, Longstaff, and Mandell (2000).

Given such a specification, the dynamics of the residual factor E_t remains the same under both the physical measure $\mathbb{P}$ and the measure $\mathbb{P}^*$. The yield curve factor F_t remains Ornstein-Uhlenbeck under $\mathbb{P}^*$, but with an adjustment to the drift term,

$$dF_t = \left(-b_\gamma - \kappa_F^* F_t\right) dt + dW_t^*, \quad \kappa_F^* = \kappa_F + \kappa_\gamma, \tag{6}$$

where $dW_t^* = dW_t + \gamma(F_t)dt$ denotes a new standard Brownian motion under measure $\mathbb{P}^*$.

Our yield curve factor specification belongs to the affine class of Duffie and Kan (1996), Duffie, Pan, and Singleton (2000), and Duffie, Filipović, and Schachermayer (2002). The fair value of the zero bond with maturity τ is exponential affine in the yield curve factor,

$$P(F_t, T) = \exp\left(-a(\tau) - b(\tau)^\top F_t\right), \tag{7}$$

[3]Since the distribution of the state vector F is symmetric, the sign of each element of b_r is not identifiable. We restrict them to be on the positive hyperplane.

where the coefficients $a(\tau)$ and $b(\tau)$ are determined by a set of ordinary differential equations. When the factors are mutually independent, as we will assume hereafter, the coefficients can be solved analytically:

$$\begin{aligned} b^i(\tau) &= \frac{b_r^i\left(1-e^{-(\kappa_F^*)^i\tau}\right)}{(\kappa_F^*)^i}, \\ a(\tau) &= a_r\tau+\sum_{i=1}^{3}\left[\frac{\left(b_r^i+2b_\gamma^i(\kappa_F^*)^i\right)}{2\left((\kappa_F^*)^i\right)^2}\left(b^i(\tau)-b_r^i\tau\right)+\frac{b^i(\tau)^2}{4(\kappa_F^*)^i}\right], \end{aligned} \tag{8}$$

where the superscript $i=1,2,3$ denotes the i-th element of a vector or the (i,i)-th element of a diagonal matrix. The fair value of the spot rate is then affine in the yield curve factors,

$$Y(F_t,T)=\frac{1}{\tau}\left(a(\tau)+b(\tau)^\top F_t\right).$$

The differences between the observed interest rates and the fair values are what we call residuals. They arise either due to the approximate nature of a model or due to temporal market imperfections. Regardless of the source, we argue that these residuals and their dynamics play important roles in the pricing of interest rate derivatives, mainly because of their direct impact on future payoffs. In our calibration exercise, we price U.S. dollar caps, which are portfolios of options written on three month LIBOR. Thus, we need to model the dynamics of the residuals on the three month LIBOR. For this purpose, we assume that the residuals on the three month LIBOR is a function of the three Gaussian residuals factors. In particular, we assume the following functional form for the loading of the three residual factors on the *observed* three-month rate,

$$y(t,t+h)=\frac{1}{h}\left(a(h)+b(h)^\top F_t+c_h^\top E_t\right), \tag{9}$$

where h denotes the exact maturity for the three month LIBOR, $c_h\in\mathbb{R}^3$ is a vector of constant parameters that determine the loading of the residual factors, and $y(t,h)$ is the continuously compounded interest rate derived from the observed LIBOR quote, based on the following relation,

$$LIBOR(h)_t=\frac{100}{h}\left(e^{hy(t,t+h)}-1\right).$$

The linear loading specification in (9) of the residual factors E on the spot rate, instead of on the LIBOR itself, is motivated partly by an analogy to the linear loading of the yield curve factors, and partly by analytical tractability in pricing caps, as we will see in the following subsection.

C. Caplet Pricing

We illustrate the impact of residual dynamics on interest rate derivatives by pricing a caplet. Each cap contract consists of a series of caplets. The payoff of the ith caplet can be written as

$$\Pi_T^i = hL\left(LIBOR(h)_T - K\right)^+$$

where h denotes the maturity of the LIBOR. It also represents the payment interval (*tenor*) of the cap contract, with $T = t + ih$. L denotes the notional amount, and K the strike rate. We assume that payment is made *in arrears*, i.e. the payment of the ith caplet is determined at time T but paid one period later at $T + h$.

Based on (2), the time-t fair value of such a caplet is given by

$$\texttt{caplet}_t^i \quad = \quad L\mathbb{E}_t^*\left[\exp\left(-\int_t^{T+h} r(s)\,ds\right) h\left(LIBOR_T(h) - K\right)^+\right].$$

Note in particular that the option (caplet) is written on the *observed* LIBOR rate, not on its fair value. The observed LIBOR rate depends upon both the yield curve factors F and the residual factors E. Writing the simply compounded LIBOR rate in terms of the continuously compounded spot rate as in (9), and by the rule of iterated expectations, we have

$$\texttt{caplet}_t^i = L\mathbb{E}_t^*\left[\exp\left(-\int_t^T r(s)\,ds\right)\left(e^{c_h^\top E_T} - (1+hK)\,e^{-a(h)-b(h)^\top F_T}\right)^+\right]. \tag{10}$$

Absent of yield curve residuals ($E_t = 0$ for all t), the caplet is equivalent to a put option on a zero coupon bond. In the presence of residuals, the caplet can be regarded as an exchange option, where one has the right to exchange the fair value of a zero coupon bond for a payoff that is a function of the residuals on that bond. The approach of DTSMs in pricing options is akin to setting $E_t = 0$ for all t, even if they are present. On the other hand, the option pricing literature often incorporates time and

maturity dependent parameters in the pricing of interest rates so that the observed yield curve is forced to the "fair value." If indeed the observed yield curve contains some residuals, these residuals will be carried over permanently into the future. This practice amounts to the following modification of (10),

$$\texttt{caplet}_t^i = L\mathbb{E}_t^* \left[\exp\left(-\int_t^T [r(s) + \mu(s)]\, ds \right) \left(1 - (1 + hK)\, e^{-a(t,h) - b(h)^\top F_T} \right)^+ \right], \tag{11}$$

where $\mu(s)$ denotes a time-inhomogeneous parameter that accommodates the observed yield curve. This adjustment not only affects the discounting, but also influence the payoff function as now $a(t,h)$ becomes time-dependent. Note the dramatic increase in dimensionality as one needs to incorporate a new parameter $\mu(t)$ for all t. Furthermore, although $\mu(t)$ is time varying and re-calibrated frequently, it is treated as a constant in pricing derivatives. The uncertainty and hence risk associated with this adjustment over time is ignored. Therefore, as argued in Dai and Singleton (2002b), such models are essentially static: they may deliver reasonable performance for cross-sectional interpolation one time at a point, but have little to say about the dynamics of the options and interest rates in the future. In contrast, under our specification in (10), we recognize the existence of potential model errors or market imperfections and explicitly account for the impacts of their dynamics on future terminal payoffs. Meanwhile, the discounting is still based on the fair value of the yield curve as in the practice of DTSMs. This treatment makes the valuation of option prices consistent with the valuation of the underlying securities.

Carrying out expectations in (10), we have,

$$\texttt{caplet}_t^i \quad = \quad LP(F_t, T+h)\left[(1 + h\mathcal{R}_t)\, \mathcal{N}(d_1) - (1 + hK)\, \mathcal{N}(d_2) \right], \tag{12}$$

where $\mathcal{N}(\cdot)$ denotes the cumulative density of a standard normal variable and the standard variables are defined as

$$d_1 = \frac{\ln(1 + h\mathcal{R}_t) / (1 + hK) + \frac{1}{2}\Sigma_t}{\sqrt{\Sigma_t}}, \quad d_2 = d_1 - \Sigma_t,$$

with $\mathcal{R}_t$ being the residual-adjusted value of the forward three-month LIBOR, defined by

$$(1 + h\mathcal{R}_t) = \frac{P(F_t, T)}{P(F_t, T+h)} \exp\left(c_h^\top \mathbb{E}_t[E_T] + \frac{1}{2} c_h^\top Var_t[E_T]\, c_h \right),$$

and Σ_t being the time-t conditional variance of $hy_T = a(h) + (h)^\top F_T + c_h^\top E_T$ under a forward measure $\mathbb{P}^T$, y_T being the future *observed* value of the three-month continuously compounded spot rate. The conditional variance Σ_t can be evaluated as

$$\Sigma_t = b(h)^\top Var_t^T[F_T]\, b(h) + c_h^\top Var_t^T[E_T]\, c_h. \tag{13}$$

The conditional mean and variance of F_T and E_T under measures and $\mathbb{P}^*$ and $\mathbb{P}^T$, as well as the derivations of the above option pricing formula, are given in Appendix A.

Equation (12) illustrates how the dynamics of the yield curve residuals influence the pricing of an interest rate caplet. The impacts come from two sources, both due to the fact that the terminal payoff of the caplet is computed based on the observed market rate, not on some model-implied fair value. First, the forward rate ($\mathcal{R}_t$) is adjusted for the expected impact of the residual dynamics. This shows up both proportionally to the caplet price and also nonlinearly in the definition of the standardized variables d_1 and d_2. Second, the conditional variance of the underlying three-month LIBOR rate in the option pricing formula Σ_t is the conditional variance of the *observed* market rate (hy_T), not that of the fair value. Thus, Σ_t captures the aggregate contribution from both the yield curve factors F and the residual factors E, as illustrated in (13).

It is important to note that the residual factors E influence the option price not only through their current, potentially nonzero, realizations (the level of E_t), but also through their conditional dynamics (e.g., conditional mean and variance of E_T). In particular, even if the current levels of the residual factors are zero, neither the residual adjustment for the forward rate nor the contribution of the residual factors to Σ_t becomes zero. In short, the current level and the future dynamics of the residual factors are equally important for the pricing of interest rate derivatives.

II. Data and Estimation

To investigate the performance of our $m+n$ model, we estimate the $3+3$ independent Gaussian affine example using a panel data of LIBOR/swap rates and cap implied volatilities. The data set is obtained from Lehman Brothers. It consists of eight years of data on (1) LIBOR rates at maturities of one, two,

three, six, and twelve months, (2) swap rates at maturities of two, three, five, seven, ten, 15, and 30 years, and (3) at-the-money caps Black implied volatilities at option maturities of one, two, three, four, five, seven, and ten years. All interest rates and interest rate options are on US dollars. The data are weekly (Wednesday) closing mid quotes from April 6th, 1994 to April 17th, 2002 (420 observations).

The U.S. dollar LIBOR rates are simply compounded interest rates, where the maturities are computed as actual over 360, starting two business days forward. The U.S. dollar swap rates have payment intervals of half years and are related to the zero prices (discount factors) by

$$SWAP(t,Nh) = 200 \times \frac{1 - p(t,t+Nh)}{\sum_{i=1}^{N} p(t,t+ih)},$$

where $h = 0.5$ is the payment interval of the swap contract and N is the swap maturity in number of payment periods. The cap contracts are on three-month LIBOR rates, with a payment interval of three month and payment is made in arrears. The strike price is set to the swap rate of the corresponding maturity. The cap implied volatility quotes are obtained under the framework of the Black model, where the LIBOR rate is assumed to follow a geometric Brownian motion. Given an implied volatility quote, the delivery price of the cap contract is computed from the Black formula.

Table I reports the summary statistics on the levels and weekly differences of LIBOR and swap rates, as well as cap implied volatilities. The average interest rates exhibit an upward sloping term structure. The standard deviation of interest rates exhibit a slight hump that peaks around six months. The interest rate levels all exhibit very high persistence. LIBOR rates also exhibit some moderate excess kurtosis. In contract, the first differences on interest rates exhibit very large excess kurtosis for LIBOR rates and moderate kurtosis for swap rates, potentially because LIBOR rates are more sensitive to the anticipation of the discontinuous fed fund movement. The cap implied volatility exhibits a hump-shaped mean term structure that peaks at three year maturity. The standard deviation of the implied volatility declines with maturity, so does the persistence. First difference in implied volatility also exhibits large excess kurtosis.

A. Quasi Maximum Likelihood (QML) with Extended Kalman Filter (EKF)

We estimate the $3+3$ Gaussian affine model using a two-stage sequential procedure. The first stage estimates the three yield curve factors using the LIBOR and swap rates. The second stage estimates the three residual factors using residuals from the yield curve and the data on cap implied volatilities. Both stages of estimation are performed with quasi maximum likelihood method jointly with extended Kalman Filter. During the first stage, the state propagation equation is controlled by the dynamics of the yield curve factors and the measurement equation is given by the fair value of LIBOR and swap rates. During the second stage, the state propagation equation is controlled by the residual factor dynamics and the measurement equation is defined by the fair value of the cap prices and the residuals on the underlying three month LIBOR rates. The extracted yield curve factors from the first stage are taken as given during the second stage estimation. Refer to Appendix B for technical details of the methodology.

Many econometric studies of affine models follow some variation of maximum likelihood estimation. While the state variables are in general not observable, they can often be directly inverted from the observed discount bonds by assuming that m of these bonds are priced *perfectly* by the m factors. The other bonds are then assumed to be priced with errors and the likelihood function can be constructed based on the conditional density of the latent variables and the pricing errors.[4] In practice, however, it is most likely that all interest rates (or bond prices) are contaminated by errors either due to model imperfections (such as misspecification or approximation error) or market imperfections (such as bid-ask spread, non-synchronous trading, misquotes, or other temporary market imperfections that have not been arbitraged away). A convenient approach to deal with these imperfections is to cast the term structure model in state space form augmented by measurement equations that relate the observed interest rates or bond prices to the underlying state variables. This leads to the application of the Kalman filtering technique, or its extended versions in dealing with nonlinear state or measurement equations. Examples of Kalman filtering in term structure model estimations include Baadsgaard, Madsen, and Nielsen (2001), Chen and Scott (2002), Duan and Simonato (1999), Duffee and Stanton (2001), and Han (2001).

[4]This "exact pricing" assumption, or some variant of it, is maintained in Chen and Scott (1993), Duffee (1999, 2002), Duffie and Singleton (1997), Honoré (1998), Liu, Longstaff, and Mandell (2000), Longstaff and Schwartz (1992), Pearson and Sun (1994), and Umantsev (2001).

The filtering technique fits naturally in our modeling framework. We first apply Kalman filtering to LIBOR and swap rates. The measurement errors on these interest rates are essentially what we call residuals on the yield curve. We make use of the residuals on the three-month LIBOR and the interest rate cap data in identifying the residual factor dynamics in a second stage estimation. We assume that the measurement errors on each series are independent but with distinct variance σ^2. Thus, we have 25 parameters for the first stage estimation: $\Theta \equiv [\sigma \in \mathbb{R}^{12+}, \kappa_F \in \mathbb{R}^{3+}, a_r \in \mathbb{R}, b_r \in \mathbb{R}^{3+}, b_\gamma \in \mathbb{R}^3, \kappa_\gamma \in \mathbb{R}^3]$. The observation equations are built on the twelve LIBOR and swap rates. For the second stage estimation, we have 14 parameters: $\Theta \equiv [\sigma \in \mathbb{R}^{8+}, \kappa_E \in \mathbb{R}^{3+}, c_h \in \mathbb{R}^3]$. The observation equations are built on the three-month LIBOR rate and the seven cap series.

Table II reports the estimates, standard errors, and p-values of the model parameters, as well as the variance of the measurement errors on each measurement equation. The estimates on most model parameters are statistically significant. Exceptions include the estimates on κ_F, which control the time series dynamics of the yield curve factors, a_r, which controls the mean level of the short rate, and $b_\gamma(2)$ and $b_\gamma(3)$, which determine the constant part of the market risk premium on the second and third yield curve factors. The large standard errors on the estimates of κ_F illustrate the inherent difficulty in estimating interest rate factor dynamics and forecasting systematic movements in interest rates. The insignificance of $b_\gamma(2)$ and $b_\gamma(3)$ may signal the absence of a constant risk premium component for the second and third yield curve factors.

III. Empirical Performance of the $3+3$ Gaussian Affine Model

Despite the simple structure, the $3+3$ independent Gaussian affine model performs well in pricing both interest rates and interest rate derivatives. The model explains over 99.5 percent of the variation in interest rates and over 95 percent of the variation in cap implied volatilities. In what follows, we investigate the sources of this performance. First, we start with the performance of the three yield curve factors in capturing the movement of the LIBOR and swap rates. We then move on to the pricing of interest rate caps with and without the residual factors. The comparison serves to gauge the significance of the yield curve residuals on the pricing of interest rate derivatives.

A. Small Residuals on the Yield Curve

Panel A of Table III reports the summary properties of the residuals (measurement errors) on the yield curve, obtained from the first stage estimation. The last column (VR) reports the percentage variation explained for each interest rate series by the three yield curve factors. It is defined as one minus the ratio of the variance of the residuals on each interest rate series to the variance of that interest rate series itself. The explained variance (VR), together with the other summary properties of the residuals, illustrates the performance of the three yield curve factors in capturing the movement of LIBOR and swap rates. In particular, the small residuals and large explained variation indicate that the three yield curve factors capture the interest rate data well. Overall, the residuals are very small, with an average mean absolute error of only four basis points. Furthermore, the last column indicates that the three yield curve factors explain over 99.5 percent of the variation of the most interest rate series.

Further analysis indicates that the measurement errors are very small for swap rates of moderate maturities (two to ten years), but are larger for LIBOR and very long maturity swap rates. For example, the mean absolute errors (MAE) are only about one basis point for two, three, and five-year swap rates, and the model fitting of the seven-year swap rate is close to perfect. On the other hand, the mean absolute errors for twelve-month LIBOR and for 30-year swap rate are about 10 basis points. The difference between short and long term swap rates may represent liquidity differences. The overall larger measurement errors on the LIBOR market may indicate some structural differences between the LIBOR and swap market that cannot be accommodated by our simple model. Furthermore, note that there exists a four basis point negative mean bias on the six month LIBOR and a ten basis point negative bias on the twelve month LIBOR. That is, both LIBOR series often stay below the model-implied yield curve. This downward bias is well recognized in the industry. For the same reason, LIBOR quotes at these two maturities are often discarded in constructing yield curves to avoid obvious kinks. Longstaff, Santa-Clara, and Schwartz (2001a) find similar market segmentations between the cap market, which is based on the LIBOR rates, and the swaption market, which is based on the swap contracts. Our results indicate that such inconsistencies in the derivatives market may actually start in the underlying interest rate market.

B. Pricing Caps with and without Interest Rate Residual Factors

Given the estimates of the three yield curve factors, we first price interest rate caps based on the model-implied yield curve, hence ignoring the potential impact of the yield curve residuals. The pricing results indicate that while the three yield curve factors can capture over 99.5 percent of variation on the yield curve, they are far from sufficient in capturing the movement of cap implied volatility. Ignoring the yield curve residuals, the three yield curve factors can only explain less than 25 percent of the aggregate variation in cap implied volatilities.

The poor performance of the yield curve factors in pricing interest rate caps is consistent with the empirical evidence in Heidari and Wu (2001), who find that three principal components from the yield curve explain over 99 percent of the variation in interest rates, but only about half of the variation in cap and swaption implied volatilities. Such evidence also highlights the limited success, and hence rare application, of structural models in pricing interest rate derivatives.[5]

The performance in pricing interest rate caps is dramatically improved when we proceed to identify the residuals on the three month LIBOR and account for their impacts on the pricing of interest rate caps. The improvement is shown in Panel B of Table III, which reports the summary statistics of the measurement errors on cap implied volatilities based on the estimated $3+3$ independent Gaussian affine model. In particular, the model can explain over 95 percent of the aggregate variation in cap implied volatilities. The model is particularly successful in explaining the variation of moderate maturity caps, with the explained percentages over 99 percent. Furthermore, except for the one-year cap series, the mean absolute errors for all other series are below half a percent, which is about the average bid-ask spread on cap implied volatility quotes.

By ignoring the interest rate residuals, we find that the three yield curve factors perform very poorly in pricing interest rate caps. This is more or less expected from the simple model structure. In particular, we assume independent factors and hence disallow factor interactions; yet, a series of research, e.g., Dai and Singleton (2000) and Leippold and Wu (2000), have demonstrated the importance of factor interactions in generating hump-shaped conditional dynamics in interest rates. Thus, by design,

[5] An exception is the recent endeavors by Singleton and Umantsev (2001), Singleton and Umantsev (2002), and Umantsev (2001).

the three-factor independent Gaussian affine model cannot generate the hump-shaped term structure often observed in interest rate caps. Furthermore, Duffee (2002) find that while Gaussian factor models with affine market price of risk do a better job of forecasting future interest rate movement, they perform poorly in capturing the conditional dynamics. In particular, a three-factor Gaussian affine model generates constant conditional variance for interest rates and hence misses the time-varying feature of conditional volatilities. Nevertheless, this design "artifact" highlights the flexibility of the $m+n$ model structure: the model prices both interest rates and interest rate derivatives well based on purely Gaussian factors and without incorporating any interactions among the $m+n$ factors.

The pricing performance on interest rates and interest rate derivatives, with and without the residual factors, is further illustrated through the sample fitting graphs in Figure 1 for some typical dates. In the Figure, circles represent market quotes (data) while lines represent model fits. The close match between the solid lines and the circles in the top panels illustrates that the three yield curve factors explain the observed yield curve well. Nevertheless, these yield curve factors perform very poorly in pricing the interest rate caps when the yield curve residuals are ignored. This is shown vividly by the dashed lines in the bottom panels. In particular, we observe how the yield curve factors cannot generate the hump-shaped term structures in cap implied volatility, observed on December 24, 1996. In contrast, the close match between the sold lines and the circles in the bottom panels illustrates that our $3+3$ Gaussian affine model can capture the different shapes of the term structure in cap implied volatilities well.

The limited success of dynamic term structure models in pricing interest rate derivatives can be attributed to their treatment of the residuals on the yield curve. In essence, such models totally ignore the residuals on the yield curve and the impacts of the dynamics of these residuals on the future payoffs of derivative contracts. Our estimation results indicate that the yield curve residuals, albeit small on the yield curve, play very important roles in the pricing of interest rate derivatives.

C. Pricing Errors on Cap Implied Volatilities

The $3+3$ Gaussian affine model explains over 95 percent of the variation in cap implied volatilities. Figure 2 plots the times series of the pricing errors of the model on the seven cap series. The model has the largest pricing errors on the one-year cap series. In particular, it seems that the model cannot

match the large variation in the short term interest rate options. Potentially, a more sophisticated model, such as one with factor interactions and/or stochastic volatility dynamics, should help alleviate such an issue.

Other than the one-year cap series, the pricing errors on all other series are remarkably small. Except for a few unusual market conditions, the pricing errors are mostly within half a percent in implied volatility, which is about the average bid-ask spread on these cap contracts. One of the unusual market conditions is the hedge fund crisis in late 1998, which has generated large pricing errors on cap implied volatilities of one, two, seven, and ten year maturities. The tragic event of September 11, 2002 has also caused large pricing errors on the cap series of almost all maturities.

IV. Dynamics of Yield Curve Residuals and Cap Pricing

We have found that small residuals on the yield curve can have significant impacts on the pricing of interest rate caps. In this section, we explore the sources of this "amplification" effect by analyzing the dynamics of the yield curve residuals. As shown by the pricing relation in (12), the dynamics of the residuals impacts the option price via its impact on the future payoff functions.

Figure 3 plots the time series of the yield curve residuals on the twelve interest rate series, extracted from the first stage estimation. From the time series plot, we observe several features of the residual dynamics that have important bearings on the pricing of interest rate derivatives. First, we observe that residuals on interest rates are a *recurring* phenomenon, rather than a one time event. If we regard the fair value of an interest rate as the equilibrium state, then the time series plots indicate that this equilibrium is not an absorbing state: the market price crosses the model implied fair value frequently without ever being absorbed by it. Therefore, in pricing options, one cannot ignore the dynamic development of these residuals, even if the current yield curve matches perfectly with the model implied fair value and the current levels of the yield curve residuals are all at zero. Hence, fitting the current yield curve perfectly through time-inhomogeneous model parameters cannot replace explicit modeling of the yield curve residual dynamics.

Second, we observe that while the residuals are fairly small on moderate maturity swap rates, both the magnitudes and the variation are significantly larger for residuals on very short-term and very long-term interest rates. For example, the residuals on the seven year swap rate are almost uniformly zero; yet, the one-month LIBOR at the very short end and the 30-year swap rate at the very long end both exhibit much larger residuals. The mean absolute errors for both series are over ten basis points, which are quite significant economically. Indeed, in some dates, the residuals are well over thirty basis points.

If we think of the moderate maturity swap rates as capturing the level of the interest rates while the two ends of the yield curve capturing the slope and curvature, these results essentially say that the three yield curve factors perform better in capturing the movement in interest rate levels than in capturing the movement of the yield curve slope and curvature. Yet, as discovered in Heidari and Wu (2001), the curvature of the yield curve contributes much more to the interest rate implied volatility movement than the interest rate level does. Thus, while the residuals on the yield curve look small over all, their impacts on the interest rate options can potentially be large due to the concentration of large residuals on the both ends of the yield curve.

Finally, all residuals on the interest rates exhibit strong mean reverting behavior. Table III reports the first order weekly autocorrelation for the residuals on each interest rate series. The average weekly autocorrelation of the pricing errors is 0.72, which corresponds to a half life of about two weeks. For comparison, Table I also reports the weekly autocorrelation of all LIBOR and swap rates, which averages around 0.984, corresponding to a half life of about ten months. Thus, the interest rate residuals are significantly more transient than the original interest rate series. They represent higher frequency signals on the yield curve.

The difference in persistence between the interest rate series and their residuals also highlights the contrast between the small percentage variance the residuals accounts for on the yield curve and their much larger impact on interest rate caps. The relative percentage variation on the yield curve is measured in terms of the *unconditional* variance, yet the impact of these residuals on option pricing is dictated by their *conditional* dynamics over the horizon of the option contract. The impact on the conditional dynamics is *amplified* relative to its contribution to the unconditional variance when the residuals are more transient than the interest rates.

To see this, let ϕ_y denote the autocorrelation over horizon τ of an observed interest rate series and let ϕ_ε denote the autocorrelation of its residual over the same horizon. Let η denote the ratio of the variance of the residual, ε, to the variance of the interest rate, y. Then, the conditional variance ratio is related to its unconditional counterpart by

$$\eta_t \equiv \frac{Var_t(\varepsilon_{t+\tau})}{Var_t(y_{t+\tau})} = \frac{Var(\varepsilon_t)(1-\phi_\varepsilon^2)}{Var(y_t)(1-\phi_y^2)} = \eta\frac{(1-\phi_\varepsilon^2)}{(1-\phi_y^2)},$$

where $Var_t(\cdot)$ denotes the conditional variance at time t and $Var(\cdot)$ without subscript denotes the unconditional variance. Thus, if the residuals are more transient than the interest rates, $\phi_\varepsilon \leq \phi_y$, the conditional variance ratio η_t will be amplified relative to the unconditional variance ratio η. Using the three month LIBOR as an example, we have $\phi_y = 0.984$ (Table I), and $\phi_\varepsilon = 0.79$ (Table III). Hence, the weekly conditional variance ratio would be amplified by close to *twelve* times compared to the unconditional variance ratio. As a result, even if the residual accounts for only a small proportion of the unconditional variation in the interest rates, they can generate large movement in the interest rate option prices.

In short, the fact that the yield curve residuals are a recurring phenomenon warrants our investigation of their dynamics and their impacts on option pricing. The fact that the residuals are larger at the two ends of the yield curve compared to the middle section of the yield curve and that the residuals are more transient than the original interest rate series explain why the residuals account for a small percentage of variation on the yield curve, but a much larger percentage on the cap implied volatility.

V. Concluding Remarks

When one calibrates a finite-dimensional model to the yield curve and the number of observations on the yield curve is more than the number of state variables, pricing errors are bound to exist due to either model approximations or market imperfections. In this paper, we propose a flexible $m+n$ model framework to accommodate the dynamics of these pricing residuals and their potential impacts on the pricing of interest rate options. Specifically, we use the first m factors to capture the systematic movement on the yield curve and the latter n factors to capture the dynamics of the yield curve residuals. Under this framework, we estimate a $3+3$ independent Gaussian affine example using a panel data of

LIBOR/swap rates and interest rate caps. The estimation results indicate that the residuals on the yield curve are overall very small, but that they have important impacts on the pricing of interest rate caps. The three yield curve factors explain over 99.5 percent of the variation on the yield curve; yet they can only explain less than 25 percent of the variation in the cap implied volatilities. Incorporating three residual factors improves the explained percentage variation to over 95 percent.

Analysis of the yield curve residuals indicates that the residuals are a recurring phenomenon and hence warrant our explicit modeling of their dynamics and their impacts on option pricing. We also find that the residuals are larger at the two ends of the yield curve compared to the middle section of the yield curve and that the residuals are more transient than the original interest rate series. Both features contribute to the result that the residuals account for a small percentage of variation on the yield curve, but a much larger percentage on the cap implied volatility.

The key contribution of our $m+n$ model specification is that it allows the dynamics of the residuals on the yield curve to be dictated by the data instead of by *a priori* assumptions. In this sense, it serves as a bridge between the dynamic term structure models and the existing option pricing literature. As a result, our model framework can consistently price both interest rates and interest rate derivatives, and can do so reasonably well on both markets, even with very simple, low-dimensional model constructs.

On top of our research agenda is to explore the application of our model structure to other financial markets. For example, the current practice of pricing mortgage backed securities is to take both the yield curve and the interest rate (implied) volatilities as given, assuming they are priced with zero errors, and then focus on the modeling and pricing of the non-economic prepayment factors. It is intriguing to see how an $m+n+p$ model structure, a direct extension of our current framework, works in pricing mortgage backed securities. In such a model structure, the m and n factors are, as we have now, the yield curve factors and the yield curve residual factors, while the last p factors could be used to capture the pricing residuals on the implied volatility surface and also the noneconomic prepayment factors. Another application is to the equity market. While equity analysts strive to identify which stock is overpriced or underpriced, the equity option pricing groups almost always ignore any useful information in such analysis and starts the pricing of options by assuming that the current stock price is at its equilibrium level. Hence, another line of future research is to apply our framework to the

consistent pricing of stock and stock options. In particular, the framework should incorporate the valuable information from stock valuations to the pricing of stock options.

Appendix A. Caplet Pricing

Given the terminal payoff of the ith caplet,

$$\Pi_T^i = hL\left(LIBOR(h)_T - K\right)^+,$$

with $T = t + ih$, its fair value can be computed via the following expectation,

$$\begin{aligned}
\texttt{caplet}_t^i &= L\mathbb{E}_t^*\left[\exp\left(-\int_t^{T+h} r(s)\,ds\right) h\left(LIBOR(h)_T - K\right)^+\right] \\
&= L\mathbb{E}_t^*\left[\exp\left(-\int_t^{T} r(s)\,ds\right)\left(e^{hy(T,T+h)} - 1 - hK\right)^+ e^{(-a(h)-b(h)F_T)}\right],
\end{aligned}$$

where the second line is obtained by representing LIBOR in terms of the continuously compounded spot rate and then by applying the rule of iterated conditional expectation. Furthermore, since

$$hy(T,T+h) - a(h) - b(h)^\top F_T = c_h^\top E_T,$$

we have

$$\texttt{caplet}^i = L\mathbb{E}_t^*\left[\exp\left(-\int_t^{T} r(s)\,ds\right)\left(e^{c_h^\top E_T} - (1+hK)\,e^{-a(h)-b(h)^\top F_T}\right)^+\right]. \tag{A1}$$

To facilitate the expectation, we perform the following measure change,

$$\texttt{caplet}_t^i = LP(F_t,T)\,\mathbb{E}_t^T\left[\left(e^{c_h^\top E_T} - (1+hK)\,e^{-a(h)-b(h)^\top F_T}\right)^+\right], \tag{A2}$$

where $\mathbb{E}_t^T[\cdot]$ denotes expectation under the fair-value *forward measure*, $\mathbb{P}^T$, defined by the following Radon-Nikodým derivative (see Musiela and Rutkowski (1997b), page 316):

$$\frac{d\mathbb{P}^T}{d\mathbb{P}^*} \equiv \frac{\exp\left(-\int_0^T r(s)\,ds\right)}{P(F_0,T)}, \quad \mathbb{P}^* - a.s.$$

Conditional on the filtration $\mathcal{F}_t$, the above Radon-Nikodým derivative satisfies, for every $t \in [0,T]$,

$$\eta_t \equiv \left.\frac{d\mathbb{P}^T}{d\mathbb{P}^*}\right|_{\mathcal{F}_t} = \frac{\exp\left(-\int_0^t r(s)\,ds\right)P(F_t,T)}{P(F_0,T)}.$$

By Itó's lemma, we have that the dynamics for the bond price $P(F_t,T)$ under measure $\mathbb{P}^*$ is given by,

$$\frac{dP(F_t,T)}{P(F_t,T)} = r(t)dt - b(T-t)^\top dW_t^*.$$

Thus, the dynamics of the density η_t can be written as,

$$\eta_t = \exp\left(-\int_0^t b(T-u)^\top dW_u^* - \frac{1}{2}\int_0^t b(T-u)^\top b(T-u)du\right),$$

and W_t^T defined by the following formula

$$W_t^T = W_t^* - \int_0^T b(T-u)du$$

follows a standard Brownian motion under the forward measure $\mathbb{P}^T$. The yield curve factor dynamics under $\mathbb{P}^T$ is then adjusted as follows,

$$dF_t = \left(-b_\gamma - b\left(T-t\right) - \kappa_F^* F_t\right)dt + dW_t^T.$$

It can be shown that under measure $\mathbb{P}^T$, and conditional on filtration $\mathcal{F}_t$, F_T is Gaussian with mean and variance given by

$$\begin{aligned}
\mathbb{E}_t^T\left[F_T\right] &= e^{-\kappa_F^*(T-t)}F_t - \frac{b_\gamma}{\kappa_F^*}\left(1-e^{-\kappa_F^*(T-t)}\right) - \frac{b_r}{2(\kappa_F^*)^2}\left(1-e^{-\kappa_F^*(T-t)}\right)^2 \\
&= e^{-\kappa_F^*(T-t)}F_t - \frac{\left(1-e^{-\kappa_F^*(T-t)}\right)}{\kappa_F^*}\left(b_\gamma + b(T-t)/2\right), \\
Var_t^T\left[F_t\right] &= \frac{1-e^{-2\kappa_F^*(T-t)}}{2\kappa_F^*}. \qquad \text{(A3)}
\end{aligned}$$

Equation (A2) can be regarded as the pricing equation for an exchange option. Directly taking expectations yields

$$\texttt{caplet}_t^i = LP\left(F_t,T\right)\mathbb{E}_t^T\left[e^{c_h^\top E_T}\right]\mathcal{N}(d_1) - (1+hK)\,\mathbb{E}_t^T\left[e^{-a(h)-b(h)^\top F_T}\right]\mathcal{N}(d_2), \qquad \text{(A4)}$$

where $\mathcal{N}(\cdot)$ denotes the cumulative density of a standard normal variable, and

$$d_1 = \frac{\ln\left(\mathbb{E}_t^T\left[e^{c_h^\top E_T}\right]/\left(1+hK\right)\mathbb{E}_t^T\left[e^{-a(h)-b(h)^\top F_T}\right]\right) + \frac{1}{2}\Sigma}{\sqrt{\Sigma_t}}, \quad d_2 = d_1 - \Sigma_t,$$

with Σ_t being the time-t conditional variance of $hy_T = a\left(h\right) + \left(h\right)^\top F_T + c_h^\top E_T$ under measure $\mathbb{P}^T$ and y_T being the future *observed* value of the three-month continuously compounded spot rate. The conditional variance Σ_t can be evaluated as

$$\Sigma_t = b\left(h\right)^\top Var_t^T\left[F_T\right]b\left(h\right) + c_h^\top Var_t^T\left[E_T\right]c_h,$$

where $Var_t^T[F_T]$ is given in (A3). The conditional mean and variance of E_T are invariant to the above measure change and are given by

$$\mathbb{E}_t[E_T] = e^{-\kappa_E(T-t)}E_t, \quad Var_t[E_T] = \frac{I - e^{-2\kappa_E(T-t)}}{2\kappa_E}.$$

Furthermore, in (A4), we have

$$\begin{aligned} \mathbb{E}_t^T\left[e^{c_h^\top E_T}\right] &= \exp\left(c_h^\top \mathbb{E}_t[E_T] + \frac{1}{2}c_h^\top Var_t[E_T]c_h\right), \\ \mathbb{E}_t^T\left[e^{-a(h)-b(h)^\top F_T}\right] &= P(F_t, T+h)/P(F_t, T). \end{aligned}$$

Rearrange, we have

$$\texttt{caplet}_t^i = LP(F_t, T+h)\left[(1+h\mathcal{R}_t)\mathcal{N}(d_1) - (1+hK)\mathcal{N}(d_2)\right],$$

where $\mathcal{R}_t$ can be regarded as the residual-adjusted value of the forward three-month LIBOR, defined by

$$(1+h\mathcal{R}_t) = \frac{P(F_t, T)}{P(F_t, T+h)}\exp\left(c_h^\top \mathbb{E}_t[E_T] + \frac{1}{2}c_h^\top Var_t[E_T]c_h\right),$$

and d_1 can be rewritten as

$$d_1 = \frac{\ln(1+h\mathcal{R}_t)/(1+hK) + \frac{1}{2}\Sigma_t}{\sqrt{\Sigma_t}}.$$

Appendix B. Extended Kalman Filter and Quasi Likelihood

Since both stages of estimation apply the same methodology. We focus on the first stage estimation as an example and then discuss necessary modifications for the second stage estimation.

The state space estimation method is based on a pair of state propagation equations and measurement equations. In our application for the first stage estimation, the state vector F propagates according to the Ornstein-Uhlenbeck process (OU) as described in (3). The measurement equation can be written as

$$y_t = h(F_t; \Theta) + \varepsilon_t, \tag{B5}$$

where y_t here denotes the observed series of LIBOR and swap rates at time t and $h(F_t; \Theta)$ denotes their corresponding fair values based on our Gaussian affine model, as a function of the state vector F_t and model parameters Θ. ε_t denotes the measurement error on the series at time t. We assume that the measurement error is independent

of the state vector and that the measurement error on each series is also mutually independent, but with distinct variance $\sigma_i^2, i = 1, \cdots, 12$.

Let $\overline{F}_t$ denote the *a priori* forecast of the state vector at time t conditional on time $t-1$ information and $\overline{V}_t$ the corresponding conditional covariance matrix. Let $\widehat{F}_t$ denote the *a posteriori* update on the time t state vector based on observations (y_t) at time t and $\widehat{V}_t$ the corresponding *a posteriori* covariance matrix. Then, based our OU state process specification, the state propagation equation is linear and Gaussian. The *a priori* update equations are:

$$\begin{aligned} \overline{F}_t &= \Phi_F \widehat{F}_{t-1}; \\ \overline{V}_t &= \Phi_F \widehat{V}_{t-1} \Phi_F^\top + Q, \end{aligned} \tag{B6}$$

where Δt denotes the discrete interval between observations, $\Phi = \exp(-\kappa_F \Delta t)$ is the autocorrelation matrix of the state vector, and Q is the covariance matrix of the state vector noise, given by

$$Q = \int_0^{\Delta t} e^{-s\left(\kappa_F + \kappa_F^\top\right)} ds = U (D)^{-1} \left(I - e^{-D\Delta t}\right) U^\top,$$

where U and D are the matrices formed by the eigenvectors and eigenvalues of $\kappa_F + \kappa_F^\top$ such that $\kappa_F + \kappa_F^\top = UDU^\top$. Finally, since we use weekly data, $\Delta t = 1/52$.

The filtering problem then consists of establishing the conditional density of the state vector F_t, conditional on the observations up to and including time t. In case of a linear measurement equation,

$$y_t = HF_t + \varepsilon_t,$$

the Kalman Filter provides the efficient *a posteriori* update on the conditional mean and variance of the state vector:

$$\begin{aligned} \bar{y}_t &= H\overline{F}_t; \\ \overline{A}_t &= H\overline{V}_t H^\top + \Sigma \\ K_t &= \overline{V}_t H \left(\overline{A}_t\right)^{-1}; \\ \widehat{F}_t &= \overline{F}_t + K_t \left(y_t - \bar{y}_t\right); \\ \widehat{P}_t &= (I - K_t H) \overline{V}_t, \end{aligned} \tag{B7}$$

where $\bar{y}_t$ and $\overline{A}_t$ are the *a priori* forecasts on the conditional mean and variance of the observed series and $\Sigma = \left(\sigma_i^2\right)$ are the covariance matrix of the measurement errors.

In our application, however, the measurement equation in (B5) is nonlinear. We hence apply the Extended Kalman Filter (EKF), which approximates the nonlinear measurement equation with a linear expansion:

$$y_t \quad \approx \quad H\left(\overline{F}_t;\Theta\right)F_t+\varepsilon_t, \tag{B8}$$

where

$$H\left(\overline{F}_t;\Theta\right)=\left.\frac{\partial h\left(\overline{F}_t;\Theta\right)}{\partial F_t}\right|_{F_t=\overline{F}_t}. \tag{B9}$$

Thus, while we still use the original pricing relation to update the conditional mean, we update the conditional variance based on this linearization. For this purpose, we need to numerically evaluate the derivative defined in (B9). We follow Norgaard, Poulsen, and Raven (2000) in updating the Cholesky factors of the covariance matrices directly.

Via the state and measurement updates, we obtain the one-period ahead forecasting error on the LIBOR and swap rates,

$$e_t=y_t-\bar{y}_t=y_t-h\left(\overline{F}_t;\Theta\right).$$

Assuming that the forecasting error is normally distributed, the quasi log-likelihood function is given by

$$\mathcal{L}\left(\mathbf{y};\Theta\right)=\sum_{t=1}^{T}l_t, \tag{B10}$$

where

$$l_t(\Theta)=-\frac{1}{2}\log\left|\overline{A}_t\right|-\frac{1}{2}\left(e_t^{\top}\left(\overline{A}_t\right)^{-1}e_t\right),$$

where the conditional mean $\bar{y}_t$ and variance $\overline{A}_t$ are given in the EFK updates in (B7).

For the second stage estimation, the state propagation is controlled by the OU process of the yield curve residual factor dynamics E_t. The measurement equations are defined by the pricing equation of the three month LIBOR and the seven cap price series. We obtain implied volatility quotes on the cap contracts but convert them into price series for the estimation. Again, the measurement errors are assumed to be independent with distinct variance.

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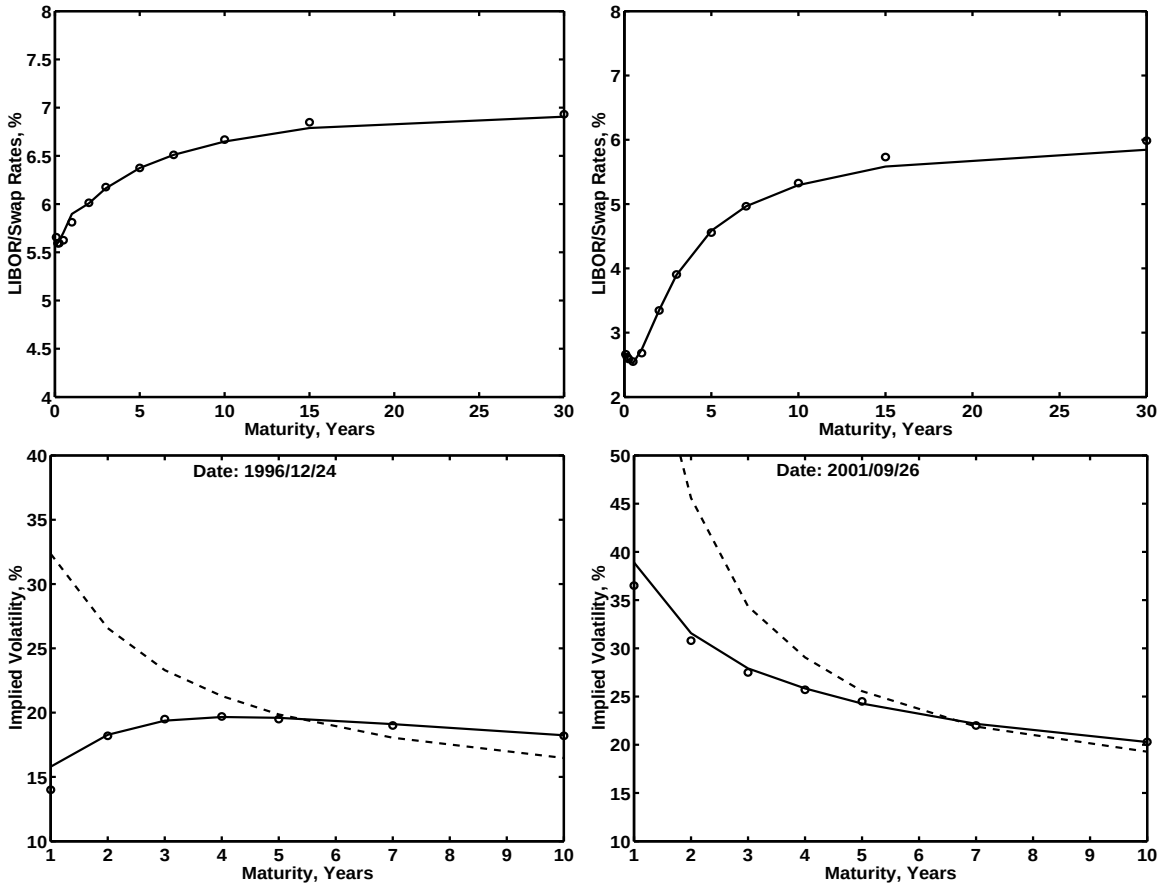

Figure 1. Pricing of Interest Rates and Interest Rate Caps

Circles denote the observed data points while lines denote the model fit. The solid lines in the top panels depict the fitting of the three yield curve factors on the LIBOR and swap rates. In the bottom panel, solid lines depict the fitting of the $3+3$ Gaussian affine model on the cap implied volatilities while the dashed lines are generated by the three yield curve factors ignoring the dynamics of yield curve residuals. Model estimates are reported in Table II.

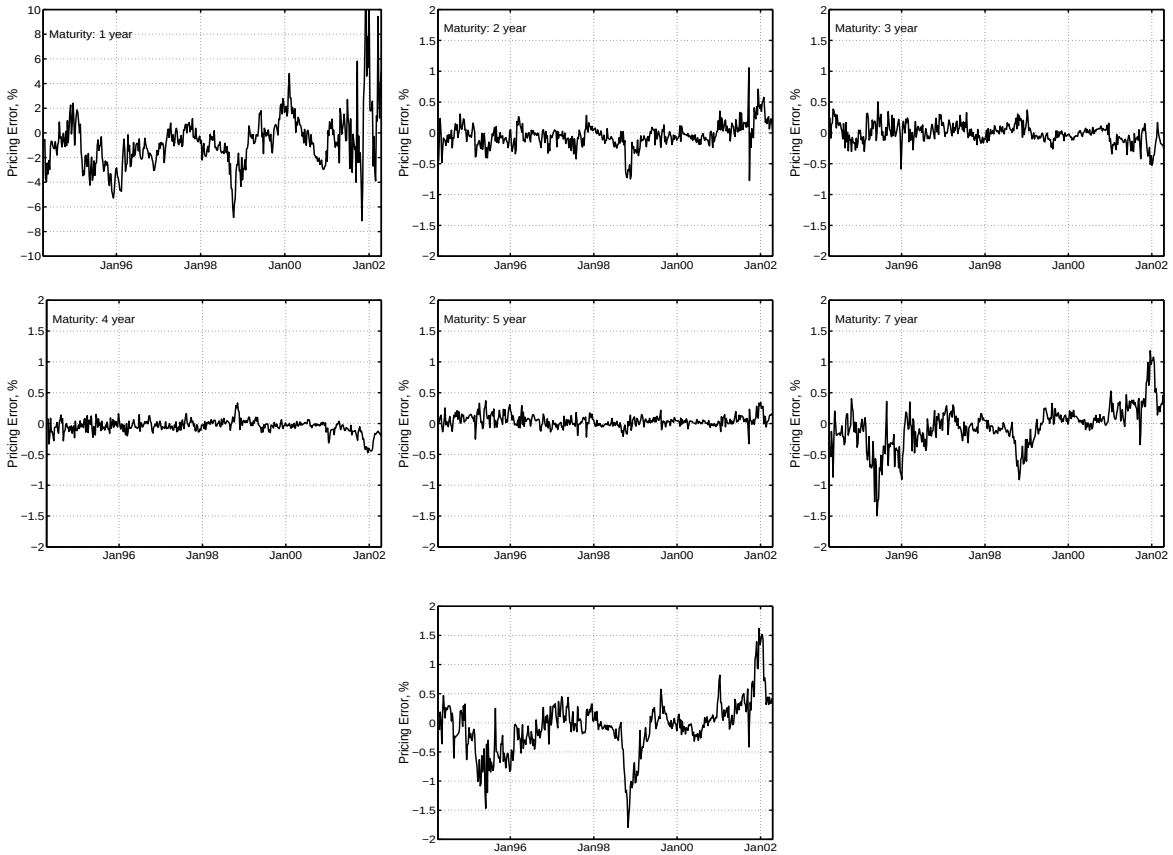

Figure 2. Pricing Errors on Cap Implied Volatilities

Graphics represent the time series of the pricing errors on cap implied volatilities. The pricing error is in percentages, defined as the difference between the observed and the model-implied cap implied volatilities. The model implied cap prices are computed via Extended Kalman Filter based on model estimates in Table II. For ease of comparison, all panels apply the same scale except for the one-year cap series, where we apply a larger scale to accommodate the larger errors.

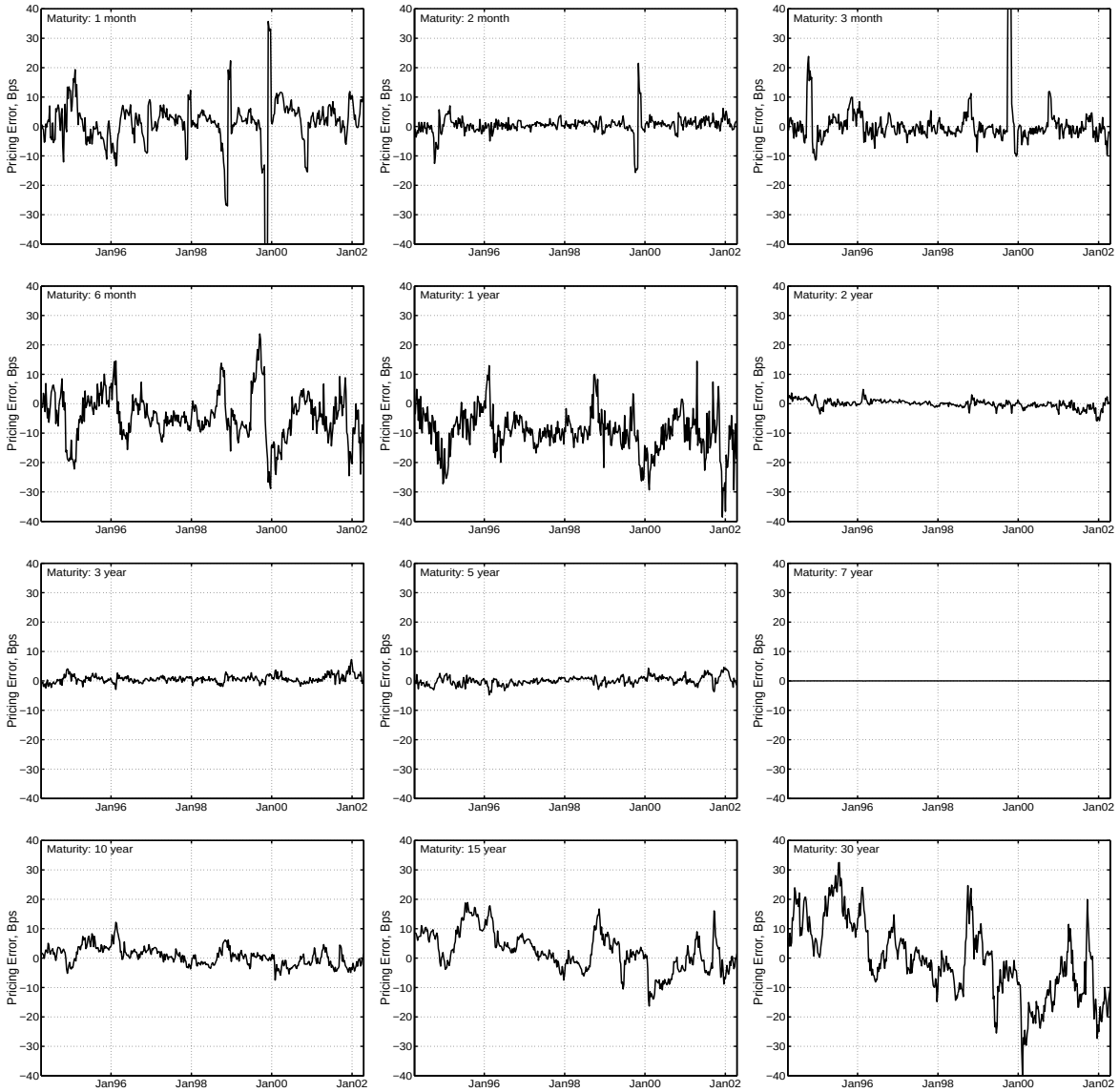

Figure 3. LIBOR/Swap Rate Residuals

Graphs represent the time series of the residuals on LIBOR and Swap Rates. The residual is in basis points, defined as the difference between the observed and the model-implied interest rates. The model implied rates are computed via Extended Kalman Filter based on model estimates in Table II. For ease of comparison, all panels apply the same scale.

Table I
Summary Statistics of Interest Rates and Implied Volatilities

Entries are summary statistics of interest rate and implied volatility data. Mean, Std, Skewness, Kurtosis, and Auto denote, respectively, the sample estimates of the mean, standard deviation, skewness, kurtosis, and first-order autocorrelation. In the maturity column, m denotes months and y denotes years. The data are weekly closing mid quotes from Lehman Brothers, from April 6th, 1994 to April 17th, 2002 (420 observations).

	Levels					Differences				
Maturity	Mean	Std	Skewness	Kurtosis	Auto	Mean	Std	Skewness	Kurtosis	Auto
					LIBOR and Swap Rates					
1 m	5.259	1.113	-1.736	2.870	0.981	-0.005	0.115	-0.606	23.073	0.081
2 m	5.296	1.123	-1.754	2.892	0.983	-0.005	0.097	-0.756	25.323	0.089
3 m	5.341	1.137	-1.750	2.854	0.984	-0.005	0.095	-1.439	18.206	0.122
6 m	5.431	1.151	-1.667	2.631	0.986	-0.005	0.098	-1.620	12.591	0.112
1 y	5.638	1.148	-1.349	1.923	0.987	-0.005	0.121	-0.685	4.094	0.051
2 y	5.908	1.012	-0.805	0.837	0.986	-0.004	0.133	0.126	1.014	0.055
3 y	6.101	0.909	-0.454	0.244	0.985	-0.004	0.135	0.156	0.864	0.030
5 y	6.333	0.806	-0.100	-0.319	0.983	-0.004	0.134	0.196	0.743	-0.012
7 y	6.480	0.762	0.048	-0.516	0.982	-0.004	0.132	0.213	0.696	-0.028
10 y	6.633	0.731	0.130	-0.607	0.981	-0.004	0.131	0.227	0.672	-0.039
15 y	6.797	0.703	0.236	-0.604	0.981	-0.004	0.125	0.178	0.619	-0.053
30 y	6.893	0.698	0.393	-0.480	0.984	-0.004	0.111	0.259	0.561	-0.019
					Cap Implied Volatility					
1 y	16.930	7.560	1.893	4.064	0.972	0.027	1.585	0.645	10.267	-0.141
2 y	18.935	5.378	1.119	1.450	0.974	0.012	1.134	0.665	5.494	-0.089
3 y	19.290	4.138	0.735	0.434	0.971	0.009	0.962	0.997	7.416	-0.094
4 y	19.162	3.461	0.533	0.079	0.970	0.006	0.818	1.054	9.859	-0.076
5 y	18.973	3.058	0.416	-0.094	0.968	0.005	0.752	0.948	10.738	-0.097
7 y	18.237	2.490	0.362	0.024	0.967	0.005	0.630	0.321	9.004	-0.076
10 y	17.339	2.071	0.415	0.193	0.961	0.003	0.572	0.196	8.718	-0.100

Table II
Parameter Estimates of the $3+3$ Independent Gaussian Affine Model

Entries report the estimates, standard errors, and p-values of the model parameters and measurement error variances. The model is estimated in two stages using quasi maximum likelihood method joint with extended Kalman filter. The data consists of weekly observations on (1) LIBOR at maturities of one, two, three, six, and twelve months, (2) swap rates at maturities of two, three, five, seven, ten, 15, and 30 years, and (3) cap prices at maturities of one, two, three, four, five, seven, and ten years. The data are closing mid quotes from April 6th, 1994 to April 17th, 2002 (420 observations). The first column denotes the model parameters with numbers in parentheses denoting the element. The fifth column denotes the time series under which the measurement equation is built upon: L denotes the LIBOR series followed by maturities in months, S denotes the swap rate series and C denotes the cap series, both followed by maturities in years. $E3$ denotes the measurement equation based on the first stage measurement error on the three month LIBOR, which achieves an almost perfect fit in the second stage and hence has an error variance estimate of close to zero. The last row reports the likelihood values for the two stages of estimation.

Model Parameters				Measurement Error Volatility, σ			
Parameters	Estimates	Std. Error	p-Value	Series	Estimates	Std. Error	p-Value
$\kappa_F(1)$	0.0182	0.2350	0.9384	$L1$	0.0961	0.0030	0.0000
$\kappa_F(2)$	0.1324	0.3098	0.6691	$L2$	0.0440	0.0025	0.0000
$\kappa_F(3)$	0.8247	0.5022	0.1005	$L3$	0.0691	0.0012	0.0000
a_r	0.0361	0.1973	0.8550	$L6$	0.0927	0.0059	0.0000
$b_r(1)$	0.0082	0.0002	0.0000	$L12$	0.1212	0.0112	0.0000
$b_r(2)$	0.0179	0.0010	0.0000	$S2$	0.0182	0.0017	0.0000
$b_r(3)$	0.0211	0.0015	0.0000	$S3$	0.0168	0.0013	0.0000
$b_\gamma(1)$	-0.1296	0.0533	0.0151	$S5$	0.0148	0.0008	0.0000
$b_\gamma(2)$	-1.9647	4.1140	0.6330	$S7$	0.0000	0.0000	0.0000
$b_\gamma(3)$	0.2508	2.4922	0.9198	$S10$	0.0310	0.0016	0.0000
$\kappa_F^*(1)$	0.0045	0.0009	0.0000	$S15$	0.0719	0.0052	0.0000
$\kappa_F^*(2)$	0.8611	0.0121	0.0000	$S30$	0.1289	0.0082	0.0000
$\kappa_F^*(3)$	2.2326	0.0646	0.0000	$E3$	0.0000	0.0000	0.0000
$\kappa_E(1)$	0.0132	0.0039	0.0008	$C1$	0.0206	0.0010	0.0000
$\kappa_E(2)$	0.6561	0.0144	0.0000	$C2$	0.0075	0.0005	0.0000
$\kappa_E(3)$	5.1150	0.2822	0.0000	$C3$	0.0111	0.0005	0.0000
$c_h(1)$	0.0011	0.0000	0.0000	$C4$	0.0128	0.0009	0.0000
$c_h(2)$	0.0017	0.0001	0.0000	$C5$	0.0183	0.0010	0.0000
$c_h(3)$	0.0029	0.0001	0.0000	$C7$	0.0758	0.0035	0.0000
				$C10$	0.1664	0.0080	0.0000
$\mathcal{L}$	$\mathcal{L}_1 = 1.0877 \times 10^4, \quad \mathcal{L}_2 = 8.6598 \times 10^3.$						

Table III
Summary Statistics of Measurement Errors on Interest Rates and Interest Rate Caps

Entries report the summary statistics of the measurement errors on LIBOR and swap rates (panel A), obtained from the first stage estimation, and on cap implied volatilities (panel B), obtained from the second stage estimation. The measurement error is defined as the difference between the observed market quotes and the model-implied fair values. The columns titled "Mean, Median, Std, MAE, Auto, Max, and Min" denote, respectively, the mean, median, standard deviation, mean absolute error, first order autocorrelation, maximum, and minimum of the measurement errors at each maturity. The last column (VR) reports the percentage variance explained for each series by the three yield curve factors in panel A and by the $3+3$ Gaussian affine model in panel B. The last row of each panel reports average statistics.

Maturity	Mean	Median	Std	MAE	Auto	Max	Min	VR
A. Errors on the Yield Curve, Basis Points								
0.1	1.30	2.01	8.75	5.57	0.73	35.73	−61.28	99.38
0.2	0.43	0.52	2.91	1.72	0.58	21.49	−15.66	99.93
0.3	0.38	−0.67	6.33	3.31	0.79	45.59	−11.45	99.69
0.5	−4.14	−4.31	8.05	7.11	0.84	23.74	−28.87	99.51
1.0	−9.57	−9.22	7.24	10.25	0.67	14.43	−38.55	99.60
2.0	−0.08	0.01	1.39	1.00	0.74	4.86	−5.86	99.98
3.0	0.65	0.57	1.27	1.05	0.65	7.28	−2.92	99.98
5.0	0.08	0.04	1.43	1.10	0.74	4.63	−4.80	99.97
7.0	−0.00	−0.00	0.01	0.01	0.11	0.04	−0.07	100.00
10.0	0.60	0.59	3.02	2.43	0.88	12.20	−7.49	99.83
15.0	2.31	1.85	6.77	5.61	0.95	18.91	−16.32	99.07
30.0	−0.51	−0.92	12.88	10.28	0.95	32.50	−39.72	96.59
Average	−0.71	−0.79	5.00	4.12	0.72	18.45	−19.42	99.46
B. Errors on Cap Implied Volatilities, %								
1	−0.78	−0.85	2.22	1.73	0.73	14.33	−7.13	91.34
2	−0.04	−0.05	0.19	0.14	0.59	1.06	−0.77	99.87
3	−0.01	−0.00	0.15	0.11	0.60	0.50	−0.59	99.87
4	−0.05	−0.03	0.11	0.08	0.65	0.33	−0.47	99.91
5	0.04	0.04	0.09	0.07	0.41	0.37	−0.33	99.91
7	−0.05	−0.02	0.34	0.24	0.84	1.18	−1.50	98.16
10	−0.07	−0.05	0.46	0.32	0.88	1.62	−1.79	95.14
Average	−0.14	−0.14	0.51	0.39	0.67	2.77	−1.80	97.74