1953

ON THE THEORY OF HEAD WAVES

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ABSTRACT

When a combined longitudinal and transverse disturbance, diverging from a localized source, strikes a plane boundary between two solid elastic media, several systems of head waves and second-order boundary waves are generated, each associated with grazing incidence of one or the other of the reflected or refracted waves. Associated with grazing incidence of $P_1\underline{P}_2$, the refracted P-wave, is the head wave system comprising $P_1P_2P_1$ (the "refracted wave" of seismic prospectors), and $P_1P_2S_1$ (a transverse head wave) in the upper medium, and $P_1P_2\underline{S}_2$ (a transverse head wave) in the lower medium. There is no boundary wave in the lower medium. These three waves, with the second-order term of $P_1\underline{p}_2$ (the first-order term is zero on the boundary) satisfy conditions of continuity of stress and displacement at the boundary. Moreover, the energy of the three head waves is derived completely from the second-order component of $P_1\underline{P}_2$, which possesses a component of energy flow normal to the boundary. The amplitudes of $P_1P_2P_1$, $P_1P_2S_1$ and $P_1P_2\underline{S}_2$ are calculated for certain cases.

REFLECTION AND REFRACTION AT A PLANE INTERFACE: FORMAL SOLUTION

In a previous paper by the author (Heelan, 1953), the mathematical form of the field radiated by a cylindrical cavity of finite length under certain prescribed conditions of stress was presented. The purpose of that study was to obtain an approximate expression for the disturbance generated by the detonation of a charge in a cylindrical shot hole. It was assumed there that the impulsive stresses acting at the source could be represented by a certain outward pressure $p(t)$, a vertical shearing stress $q(t)$, and a horizontal shearing stress $s(t)$. As far as the following work is concerned, however, the source of the radiating disturbance can be taken to be any localized disturbance in the upper medium radiating P, SV, and SH waves of which the horizontal and vertical particle displacements (predominant terms only) can be respectively expressed in the following forms: for P,

$$\begin{bmatrix} u_P \\ w_P \end{bmatrix} = \begin{bmatrix} \frac{F_1(\phi)}{R} & \frac{d}{dt}\{p(t - R/V)\} \end{bmatrix} \begin{bmatrix} \sin\phi \\ -\cos\phi \end{bmatrix}; \quad (1)$$

for SV,

$$\begin{bmatrix} u_{SV} \\ w_{SV} \end{bmatrix} = \begin{bmatrix} \frac{F_2(\phi)}{R} & \frac{d}{dt}\{p(t - R/v)\} \end{bmatrix} \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}; \quad (2)$$

and for SH,

$$v_{SH} = \frac{K(\phi)}{R}\frac{d}{dt}\{s(t - R/v)\}, \quad (3)$$

where (see Figure 1)

* Manuscript received by the Editor October 3, 1952. This paper is based on a portion of a doctoral dissertation written by the author at Saint Louis University under the direction of the Reverend James B. Macelwane, S.J.

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$$R^2 = r^2 + (d - Z)^2, \qquad \tan\phi = r/(d - Z).$$

For the particular case of a small cylindrical source of the type considered in the previous paper, $p(t)$, as we have shown, represents the outward lateral pressure at the source, and $s(t)$ the horizontal shearing stress. The vertical shearing stress, $q(t)$ of the previous paper, is assumed to be zero. In this case,

$$F_1(\phi) = \Delta(1 - [2v^2\cos^2\phi]/V^2)/4\pi\mu V$$
$$F_2(\phi) = \Delta\sin 2\phi/4\pi\mu v$$
$$K(\phi) = \Delta\sin\phi/4\pi\mu v,$$

where Δ = volume of the cylindrical source,
μ = rigidity of the medium,
V, v = velocities of P and S waves respectively.

We now proceed to examine how the incident radiation is modified by the presence of a plane discontinuity in the medium.

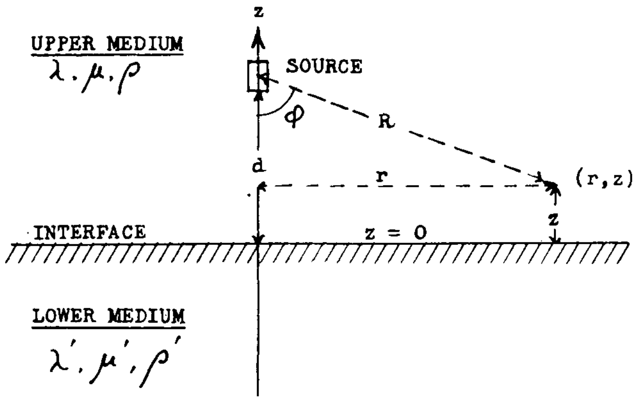

FIG. 1. Geometry of system.

It is supposed that the center of the disturbance is located at the point $z = d$, $r = 0$ in a medium of elastic constants λ and μ and of density ρ filling the half-space $z > 0$. The half-space $z < 0$ is occupied by a medium of elastic constants λ' and μ' and of density ρ' (see Figure 1).

The primary incident radiation generated by the source has the form of the three auxiliary wave functions,

$$\Phi_0 = \int_0^\infty e^{ikVt}dk \int_C f_0 H_0^{(1)}(\sigma r) e^{\alpha(z-d)} d\sigma$$

$$\Theta_0 = \int_0^\infty e^{ikVt}dk \int_C g_0 H_0^{(1)}(\sigma r) e^{\beta(z-d)} d\sigma$$

$$\chi_0 = \int_0^\infty e^{ikVt}dk \int_C n_0 H_0^{(1)}(\sigma r) e^{\beta(z-d)} d\sigma,$$

where $\alpha=(\sigma^2-k^2)^{1/2}$, $\beta=(\sigma^2-h^2)^{1/2}$, $kV=hv$, and C is a loop $(\infty i, -k, -h, \infty i)$ where arg. σ = arg. α = arg. $\beta=\pi/2$ initially, and $2\pi \geqq$ arg. $\sigma \geqq 0$ on the path. Φ_0 represents the incident longitudinal wave, Θ_0 the incident SV wave, and χ_0 the incident SH wave. These integrals are assumed to give waves of the type represented by equations (1) at large distances from the source. For the case of the small cylindrical source treated in the previous paper, the functionals f_0, g_0 and n_0 assume the following forms at large distances from the source:

$$f_0 = p_1(k)\Delta\sigma(2\sigma^2/h^2 + 1 - 2v^2/V^2)/8\pi\mu(\sigma^2 - k^2)^{1/2},$$

$$g_0 = p_1(k)\Delta\sigma/4\pi\mu h^2,$$

$$n_0 = s_1(k)\Delta\sigma/4\pi\mu(\sigma^2 - h^2)^{1/2},$$

where $p_1(k)$ and $s_1(k)$ satisfy the relationships

$$p(t) = \mathrm{Rl} \int_0^\infty p_1(k) \exp(ikVt)dk$$

and

$$s(t) = \mathrm{Rl} \int_0^\infty s_1(k) \exp(ikVt)dk,$$

Rl designating the real portion of the integrals.

The expressions for the waves themselves are given in equations (1) to (3).

Let Φ, Θ, χ be the auxiliary wave functions[1] of the reflected longitudinal, SV and SH disturbances respectively, and Φ', Θ', χ' the corresponding functions for the transmitted disturbance. These must satisfy the following equations.

$$\frac{\partial^2\Phi}{\partial t^2} - V^2\nabla^2\Phi = 0; \qquad \frac{\partial^2\Phi'}{\partial t^2} - V'^2\nabla^2\Phi' = 0$$

$$\frac{\partial^2\Theta}{\partial t^2} - v^2\nabla^2\Theta = 0; \qquad \frac{\partial^2\Theta'}{\partial t^2} - v'^2\nabla^2\Theta' = 0$$

$$\frac{\partial^2\chi}{\partial t^2} - v^2\nabla^2\chi = 0; \qquad \frac{\partial^2\chi'}{\partial t^2} - v'^2\nabla^2\chi' = 0$$

where $V=(\lambda+2\mu)^{1/2}/\rho^{1/2}$, $v=\mu^{1/2}/\rho^{1/2}$, $V'=(\lambda'+2\mu')^{1/2}/\rho'^{1/2}$, $v'=\mu'^{1/2}/\rho'^{1/2}$.

It is now assumed that Φ, Θ, χ, Φ', Θ', and χ' can be expressed as integrals in the following way:

[1] The particle displacements (u, v, w) in the directions of r, θ, z respectively increasing, are given by the formulas:

$$u = \frac{\partial\Phi}{\partial r} - \frac{\partial^2\Theta}{\partial r\partial z}, \qquad w = \frac{\partial\Phi}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Theta}{\partial r}\right), \qquad v = \frac{\partial\chi}{\partial r}.$$

$$\begin{aligned}
\Phi &= \int_0^\infty e^{ikVt}dk \int_C f_1 H_0^{(1)}(\sigma r) e^{-\alpha z} d\sigma \\
\Theta &= \int_0^\infty e^{ikVt}dk \int_C g_1 H_0^{(1)}(\sigma r) e^{-\beta z} d\sigma \\
\chi &= \int_0^\infty e^{ikVt}dk \int_C n_1 H_0^{(1)}(\sigma r) e^{-\beta z} d\sigma \\
\Phi' &= \int_0^\infty e^{ikVt}dk \int_C f' H_0^{(1)}(\sigma r) e^{-\alpha' z} d\sigma \\
\Theta' &= \int_0^\infty e^{ikVt}dk \int_C g' H_0^{(1)}(\sigma r) e^{-\beta' z} d\sigma \\
\chi' &= \int_0^\infty e^{ikVt}dk \int_C n' H_0^{(1)}(\sigma r) e^{-\beta' z} d\sigma
\end{aligned} \tag{4}$$

where $kV = hv = k'V' = h'v'$, and C is now an enlarged contour including the additional branch points $-k'$ and $-h'$, and any other singularity of the integrands that provides a physically interpretable result.[2]

Expressing the continuity of particle displacement and stress across the plane $z = 0$, six linear equations are obtained for the six functional unknowns. Solving these equations, it is found that

$$\left.\begin{aligned}
f_1 &= [f_0 D_1 \exp(-\alpha d) + g_0 D_2 \exp(-\beta d)]/D \\
g_1 &= [f_0 D_3 \exp(-\alpha d) + g_0 D_4 \exp(-\beta d)]/D \\
n_1 &= n_0(\mu\beta - \mu'\beta')(\mu\beta + \mu'\beta')^{-1} \exp(-\beta d) \\
f' &= [f_0 D_1' \exp(-\alpha d) + g_0 D_2' \exp(-\beta d)]/D \\
g' &= [f_0 D_3' \exp(-\alpha d) + g_0 D_4' \exp(-\beta d)]/D \\
n' &= 2n_0\mu\beta(\mu\beta + \mu'\beta')^{-1} \exp(-\beta d)
\end{aligned}\right\} \tag{5}$$

where the coefficients D, D_1, etc. are given by the following sets of equations, in which $\xi = 2\sigma^2 - h^2$ and $\xi' = 2\sigma^2 - h'^2$:

$$D = \alpha' P + Q \tag{6}$$

where

$$P = 4\alpha\beta\beta'\sigma^2(\mu' - \mu)^2 - \beta(\mu\xi - 2\mu'\sigma^2)^2 - \mu\mu' h^2 h'^2 \beta$$

$$Q = -\alpha\beta(\mu'\xi' - 2\mu\sigma^2)^2 - \mu\mu' h^2 h'^2 \alpha\beta' + \sigma^2(\mu'\xi' - \mu\xi)^2$$

$$D_1 = \alpha' P_1 + Q_1 \tag{7}$$

[2] The wave system associated with the singularity must be of divergent type and finite at infinty, *cf.* Sommerfeld, A. (1912).

where

$$P_1 = 4\alpha\beta\beta'\sigma^2(\mu' - \mu)^2 + \beta'(\mu\xi - 2\mu'\sigma^2)^2 + \mu\mu' h^2 h'^2 \beta$$

$$Q_1 = -\alpha\beta(\mu'\xi' - 2\mu\sigma^2)^2 - \mu\mu' h^2 h'^2 \alpha\beta' - \sigma^2(\mu'\xi' - \mu\xi)^2$$

$$D_2 = \beta' P_2 + Q_2 \tag{8}$$

where

$$P_2 = 4\alpha'\beta\sigma^2(\mu' - \mu)(\mu\xi - 2\mu'\sigma^2)$$

$$Q_2 = 2\beta\sigma^2(\mu'\xi' - \mu\xi)(\mu'\xi' - 2\mu\sigma^2),$$

$$D_3 = \alpha' P_3 + Q_3 \tag{9}$$

where

$$P_3 = 4\alpha\beta'(\mu' - \mu)(\mu\xi - 2\mu'\sigma_2)$$

$$Q_3 = 2\alpha(\mu'\xi' - \mu\xi)(\mu'\xi - 2\mu\sigma^2)$$

$$D_4 = \beta' P_4 + Q_4 \tag{10}$$

where

$$P_4 = 4\alpha\alpha'\beta\sigma^2(\mu' - \mu)^2 + \alpha'(\mu\xi - 2\mu'\sigma^2)^2 + \mu\mu' h^2 h'^2 \alpha$$

$$Q_4 = -\mu\mu' h^2 h'^2 \alpha'\beta - \alpha\beta(\mu'\xi' - 2\mu\sigma^2)^2 - \sigma^2(\mu'\xi - \mu\xi)^2,$$

$$D_1' = 2\mu h^2 \alpha\beta(\mu'\xi' - 2\mu\sigma^2) + 2\mu h^2 \alpha\beta'(\mu\xi - 2\mu'\sigma^2) \tag{11}$$

$$D_2' = \beta' P_2' + Q_2' \tag{12}$$

where

$$P_2' = 4\mu h^2 \alpha\beta\sigma^2(\mu' - \mu)$$

$$Q_2' = -2\mu h^2 \beta\sigma^2(\mu'\xi' - \mu\xi)$$

$$D_3' = \alpha' P_3' + Q_3' \tag{13}$$

where

$$P_3' = 4\mu h^2 \alpha\beta(\mu' - \mu)$$

$$Q_3' = -2\mu h^2 \alpha(\mu'\xi' - \mu\xi)$$

and

$$D_4' = 2\mu h^2 \alpha\beta(\mu'\xi' - 2\mu\sigma^2) + 2\mu h^2 \alpha'\beta(\mu\xi - 2\mu'\sigma^2). \tag{14}$$

When the functionals (equations (5)) are substituted into their respective integrals, it is seen that the reflected and transmitted longitudinal disturbances are each composed of two parts, one involving f_0 and consequently the incident P wave, and the other g_0 and the incident SV wave. The reflected and transmitted SV disturbances are similarly composed. Only SH acts independently and is reflected and transmitted wholly without change of type.

Reflected Longitudinal Disturbance.—Changing over to actual particle displacements, the reflected longitudinal disturbance in the upper-medium is found to be the sum of: 1. (u_1, w_1) which yields, as we shall see later, the principal reflected PP wave as its principal part, and 2. (u_2, w_2) which yields SP as its principal part, where

$$\left.\begin{aligned} u_1 &= -\int_C \frac{\sigma f_0 D_1}{D} H_1^{(1)}(\sigma r) e^{-\alpha(z+d)} d\sigma \\ w_1 &= -\int_C \frac{\alpha f_0 D_1}{D} H_0^{(1)}(\sigma r) e^{-\alpha(z+d)} d\sigma \end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned} u_2 &= -\int_C \frac{\sigma g_0 D_2}{D} H_1^{(1)}(\sigma r) e^{-\alpha z-\beta d} d\sigma \\ w_2 &= -\int_C \frac{\alpha g_0 D_2}{D} H_0^{(1)}(\sigma r) e^{-\alpha z-\beta d} d\sigma \end{aligned}\right\}. \tag{16}$$

Reflected SV Disturbance.—This is composed of: 1. (u_3, w_3) which yields PS, and 2. (u_4, w_4) which yields SS. Here

$$\left.\begin{aligned} u_3 &= -\int_C \frac{\sigma\beta f_0 D_3}{D} H_1^{(1)}(\sigma r) e^{-\beta z-\alpha d} d\sigma \\ w_3 &= -\int_C \frac{\sigma^2 f_0 D_3}{D} H_0^{(1)}(\sigma r) e^{-\beta z-\alpha d} d\sigma \end{aligned}\right\} \tag{17}$$

$$\left.\begin{aligned} u_4 &= -\int_C \frac{\sigma\beta g_0 D_4}{D} H_1^{(1)}(\sigma r) e^{-\beta(z+d)} d\sigma \\ w_4 &= -\int_C \frac{\sigma^2 g_0 D_4}{D} H_0^{(1)}(\sigma r) e^{-\beta(z+d)} d\sigma \end{aligned}\right\}. \tag{18}$$

Reflected SH Disturbance.—This has only one term v_1 which corresponds to SS (horizontally polarized). Here,

$$v_1 = -\int_C \frac{\mu\beta - \mu'\beta'}{\mu\beta + \mu'\beta'} \sigma q_0 H_1^{(1)}(\sigma r) e^{-\beta(z+d)}. \tag{19}$$

Transmitted Longitudinal Disturbance.—This is composed of two parts, 1. (u_1', w_1') which gives $P\underline{P}$,[3] and 2. (u_2', w_2') which gives $S\underline{P}$. Here,

[3] The underbar denotes a path in the lower medium.

$$\left.\begin{aligned} u_1' &= -\int_C \frac{\sigma f_0 D_1'}{D} H_1^{(1)}(\sigma r) e^{\alpha' z - \alpha d} d\sigma \\ w_1' &= \int_C \frac{\alpha' f_0 D_1'}{D} H_0^{(1)}(\sigma r) e^{\alpha' z - \alpha d} d\sigma \end{aligned}\right\} \tag{20}$$

$$\left.\begin{aligned} u_2' &= -\int_C \frac{\sigma g_0 D_2'}{D} H_1^{(1)}(\sigma r) e^{\alpha' z - \beta d} d\sigma \\ w_2' &= \int_C \frac{\alpha' g_0 D_2'}{D} H_0^{(1)}(\sigma r) e^{\alpha' z - \beta d} d\sigma \end{aligned}\right\}. \tag{21}$$

Transmitted SV Disturbance.—This likewise, is composed of two parts: 1. (u_3', w_3') which gives $P\underline{S}$, and 2. (u_4', w_4') which gives $S\underline{S}$. Here,

$$\left.\begin{aligned} u_3' &= -\int_C \frac{\sigma \beta' f_0 D_3'}{D} H_1^{(1)}(\sigma r) e^{\beta' z - \alpha d} d\sigma \\ w_3' &= -\int_C \frac{\sigma^2 f_0 D_3'}{D} H_0^{(1)}(\sigma r) e^{\beta' z - \alpha d} d\sigma \end{aligned}\right\} \tag{22}$$

$$\left.\begin{aligned} u_4' &= \int_C \frac{\sigma \beta' g_0 D_4'}{D} H_1^{(1)}(\sigma r) e^{\beta' z - \beta d} d\sigma \\ w_4' &= -\int_C \frac{\sigma^2 g_0 D_4'}{D} H_0^{(1)}(\sigma r) e^{\beta' z - \beta d} d\sigma \end{aligned}\right\}. \tag{23}$$

Transmitted SH Disturbance.—This has only one term v', which corresponds to $S\underline{S}$ (horizontally polarized). Here,

$$v' = -\int_C \frac{2\mu\sigma\beta n_0}{\mu\beta + \mu'\beta'} H_1^{(1)}(\sigma r) e^{\beta' z - \beta d} d\sigma. \tag{24}$$

In each of the preceding formulas, the operation $\int_0^\infty \cdots e^{ikVt}\, dk$ (real part) has been omitted merely for convenience in writing the expressions. It is understood to apply to each of the integrals numbered (15) to (24).

The preceding formulas comprise the complete formal solution of the problem of reflection and transmission of a given disturbance at a plane interface separating two solid media. In the next paper, it will be shown that each of these integrals, when evaluated, yields a number of terms, some of the first order, delineating the major portion of the effect under ordinary conditions, and others of the second-order, among which are included many forms of head waves. These head waves will be the subject of the following two sections.

NATURE OF WAVE SYSTEMS GENERATED BY REFLECTION AND REFRACTION AT A PLANE BOUNDARY

The expressions (15) to (24) which describe the particle displacements in the two media, constitute the formal solution of the problem of reflection and refraction of a disturbance at a plane interface between two media. As they stand, however, they do not yield much information about the nature of the separate wave systems generated at the boundary. In the neighborhood of the source, a quantitative description of the disturbance would require a laborious numerical integration. For most practical purposes it is sufficient to consider what happens at distances from the source sufficiently large to justify the use of asymptotic expansions in inverse powers of the distance from the source.

The general method to be used in obtaining asymptotic expansions involves a deformation of the path of integration C, so that the predominant terms can be procured by successive applications of Watson's Lemma (Copson, 1935) to segments of the path. This method has been employed successfully in similar problems by Nakano, Sezawa, Kanai, Nishimura, Sakai, Scholte and others (see, for example Nakano, 1925).

Consider the first of the integrals numbered (15), namely,

$$u_1 = -\int_C \frac{\sigma D_1 f_0}{D} H_1^{(1)}(\sigma r) e^{-(z+d)\alpha} d\sigma.$$

When $|\sigma r| \gg 0$ at all points on the path C, it is possible to replace the Hankel function by its asymptotic expansion,

$$\sqrt{\frac{2}{\pi\sigma r}}\, e^{i\sigma r - 3\pi i/4}.$$

Putting $z+d = R_1 \cos \epsilon$, $r = R_1 \sin \epsilon$ (see Figure 2) and $m_1(\sigma) = \alpha \cos \epsilon - i\sigma \sin \epsilon$, the integral then reduces to

$$-\sqrt{\frac{2}{\pi r}} \int_C \frac{\sigma^{1/2} D_1 f_0}{D} e^{-R_1 m_1(\sigma) - 3\pi i/4} d\sigma.$$

This form can be handled effectively by means of Debye's Method of Steepest Descent (see Copson, 1935). The contour is deformed continuously into the path of steepest descent through an appropriate saddle point of the real part of the exponent $m_1(\sigma)$. If some of the singularities lie outside this path, loops must be added connecting these singularities with the new curve. These loops begin and end on the path of steepest descent. With this provision, the new path of integration is equivalent to the old.

The appropriate saddle point in this case is $\sigma_0 = -k \sin \epsilon$ $(\alpha = ik \cos \epsilon)$. The path of steepest descent is a curve which crosses the real axis of the σ-plane

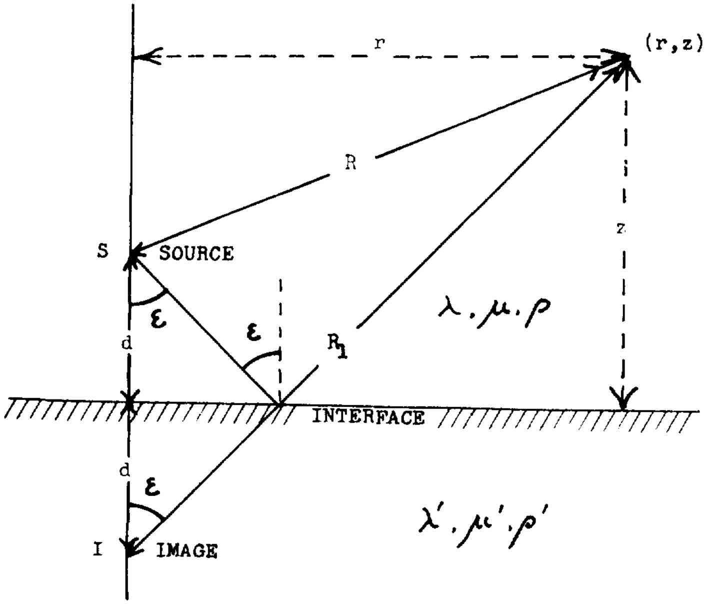

FIG. 2. Direct and reflected wave paths.

at points $-k \sin \epsilon$ and $-k/\sin \epsilon$, with asymptotes making angles of ϵ and $\pi-\epsilon$ with the real σ-axis (see Fig. 3). The singularities of the integrand are the branch points $-k$, $-h$, $-k'$, $-h'$ and the roots of $D=0$. Leaving out of consideration the poles, which are associated with Stoneley and pseudo-Rayleigh type waves along the boundary, the branch points are all distributed along the negative real axis, and depending upon the value of sin ϵ lie inside or outside the path of steep-

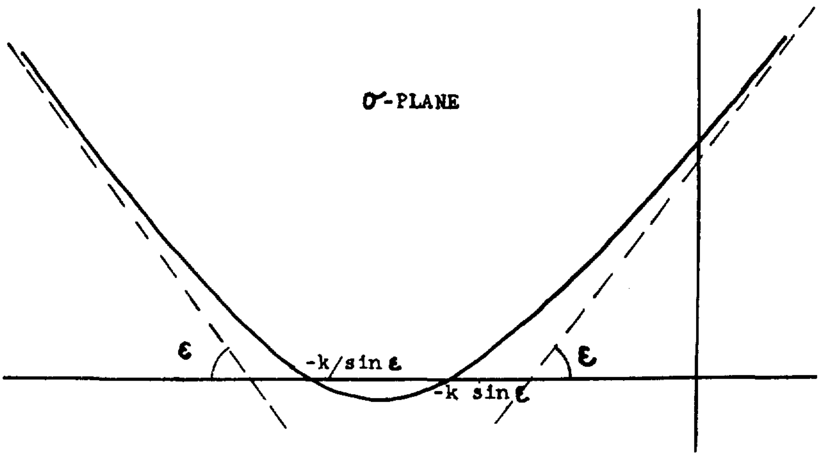

FIG. 3. Path of steepest descent.

est descent. For example, the condition that $-k'$ lie outside the path of steepest descent is that $-k \sin \epsilon < -k'$, or $\sin \epsilon > k'/k = V/V' = \sin i_1$. In this case, in order to preserve the equivalence of contours between the new and the old, a loop must be added connecting $-k'$ to the path of steepest descent.

Making the transformation, $\xi = m_1(\sigma) - m_1(\sigma_0)$ and applying Watson's Lemma, the contribution of the path of steepest descent is found to be

$$\frac{2k \cos \epsilon}{R_1} \left[f_0(\sigma) D_1/D\right]_{\sigma_0} e^{-ikR_1}$$

where the bracketed quantity is evaluated at the point $\sigma = \sigma_0 = -k \sin \epsilon$. The expression $[D_1/D]_{\sigma_0}$ is identical with the reflection coefficient for a P-wave incident at angle ϵ and reflected as a P-wave.[4] Putting

$$[D_1/D]_{\sigma_0} = A(\epsilon) = A'(\epsilon) + iA''(\epsilon),$$

it is found that $A''(\epsilon) = 0$, unless $\epsilon > \text{arc} \sin V/V'$. Applying $\int_0^\infty \cdots e^{ikVt} dk$(real part) as an operator, the principal part of this wave emerges as

$$\begin{bmatrix} u_1 \\ w_1 \end{bmatrix} = \left[\frac{F_1(\epsilon)}{R_1}\left(A'(\epsilon) \frac{d}{dt}\{p(t - R_1/V)\} - A''(\epsilon) \frac{d}{dt}\{p_1(t - R_1/V)\}\right)\right] \begin{bmatrix} \sin \epsilon \\ \cos \epsilon \end{bmatrix}$$

where $F_1(\epsilon)$ is related to the amplitude of the incident P-wave as shown in equations (1) and where

$$\frac{d}{dt}\{p(t - R_1/V)\} = \text{Rl} \int_0^\infty ikVp_1(k) e^{ikV(t - R_1/V)} dk$$

$$\frac{d}{dt}\{p_1(t - R_1/V)\} = \text{Im} \int_0^\infty ikVp_1(k) e^{ikV(t - R_1/V)} dk.$$

The phase retardation R_1/V shows that this represents the reflected P-wave, P_1P_1.[5]

Contribution of the branch point $-k'$: It has been shown that when $\epsilon > \text{arc} \sin V/V'$, an additional contribution is made by integration around the loop that connects the branch point $-k'$ with the path of steepest descent. Choosing for this loop the path defined by keeping the imaginary part of $[m_1(\sigma) - m_1(-k')]$ zero, and taking this path twice about $-k'$ (see Fig. 4) and connecting it to

[4] For numerical values, see, for example, Slichter and Gabriel (1933) or Muskat and Meres (1940).

[5] The individual letters, in the usual convention, represent segments of the ray path (real or hypothetical). The subscript refers to the velocity with which the segment is traversed; one, for velocities characteristic of the upper medium, and two, for those characteristic of the lower medium.

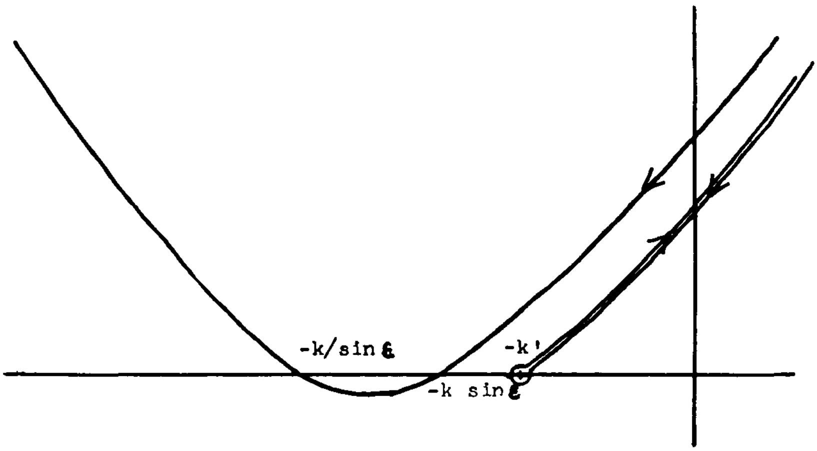

FIG. 4. Path of steepest descent with loop around the branch point $-k'$

the path of steepest descent at infinity, it is possible to change to the *real* variable $\xi = m_1(\sigma) - m_1(-k')$ and apply Watson's Lemma to the resulting integral. The principal part of this integration turns out to be

$$\begin{bmatrix} u_1(k') \\ w_1(k') \end{bmatrix} = \left[\frac{XF_1(i)p(\theta_1)}{r^{1/2}L_1^{3/2}}\right]\begin{bmatrix} \sin i_1 \\ \cos i_1 \end{bmatrix} \qquad \begin{matrix}(25a)\\(25b)\end{matrix}$$

where

$i_1 = \arcsin V/V'$, $L_1 = r - (z + d) \tan i_1$, $\theta_1 = t - [(z + d) \cos i_1]/V - r/V'$,

and

$$X = \left[\frac{i\sigma(P_1Q - Q_1P)}{Q^2 \cos i_1}\right]_{-k'}.$$

The phase retardation, $[(z+d) \cos i_1]/V + r/V'$, corresponds, as Muskat (1933) has shown, to the time taken for a wave to travel from the source to the point (r, z) by a path composed of the three segments SA, AB and BC shown in Figure 5, where SA is traversed with velocity V, AB with velocity V', and BC with velocity V. The vibration of the particle is longitudinal to the ray BC. This wave evidently corresponds to the head wave or "refracted wave" used in seismic prospecting. It may be denoted after Muskat (1933) by the three hypothetical segments of its path, namely $P_1P_2P_1$, where the subscripts refer to the velocities with which the segments are traversed.

The amplitude (equations 25a, 25b) of $P_1P_2P_1$ is the product of several factors:

1. $F_1(i_1)$ shows that the amplitude is a function of the amplitude of the *incident critical ray.*

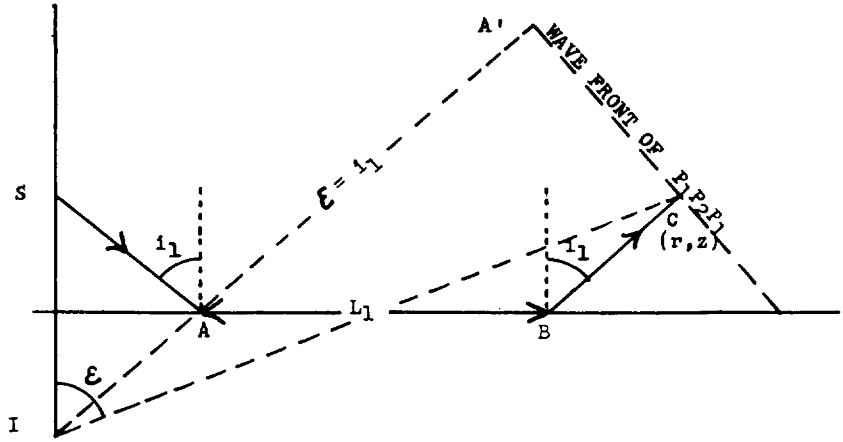

FIG. 5. Wave front of $P_1P_2P_1$.

2. $p(\theta_1)$, the time dependent factor, is related to the time dependent term of the incident radiation,[6] *viz.* $(d/dt)\{p(t-R/V)\}$ as a function is related to its time derivative. For the case of a small cylindrical source, $p(\theta_1)$ is identical with the impressed lateral force at the source, though retarded in phase by an amount corresponding to the travel time along its hypothetical path. (See Heelan, 1953.)

3. The presence in the denominator of $L_1 = AB$ (in Fig. 5), which is zero along the boundary AA' (or $\epsilon = i_1$) of the domain of existence of $P_1P_2P_1$, means that expressions (25a) and (25b) are not valid on AA'.

For points on $AA'(\epsilon = i_1 = \text{arc sin } V/V')$, the path of steepest descent passes through $-k'$, which is a singularity of the integrand, and the path of integration must then be indented by a small semi-circle so as to pass to the right of this point. The resulting integration gives an asymptotic series in descending *fourth* powers of R_1, the first term of which is identical with the principal term of P_1P_1.

The other branch points, $-h$ and $-h'$, also make their contributions to the value of the integral, but $-k$, which always lies within the path of steepest descent, contributes nothing. In the first case, if $-h < -k/\sin \epsilon$, i.e., $\epsilon > \text{arc sin } v/V$, a type of second-order boundary wave is obtained which we denote by $(S_1)_1$.[7] In the second case, if $-k \sin \epsilon < -h'$, i.e. $\epsilon > \text{arcsin } V/v'$, a head wave $P_1S_2P_1$ is obtained. If, on the other hand, $-k/\sin \epsilon > -h'$, i.e. $\epsilon > \text{arcsin } v'/V$, a type of second-order boundary wave, which we denote by $(S_2)_1$, is obtained. These results, with the corresponding results for the other integrals (15) to (24), are summarized in Tables I to IV that follow. The column headings denote the particular point in the σ-plane with which the particular wave form is associated.

Thus, beside the first order reflected and refracted waves, and the first order

[6] *Cf.* equation (1).

[7] Parentheses denote a boundary wave, i.e., a wave with amplitude diminishing exponentially with distance from the boundary.

TABLE I

WAVE SYSTEMS IN THE UPPER MEDIUM
P, SV COMPONENTS

	First Order Waves		Second Order Waves				
	Saddle Point	Roots $D=0$	$-k'$	$-k$	$-h'$	$-h$	Wave Type
u_1, w_1	P_1P_1	Stoneley and pseudo-Rayleigh Waves	$P_1P_2P_1$	—	$P_1S_2P_1$[a] $(S_2)_1$[b]	$(S_1)_1$	irrotational
u_2, w_2	S_1P_1		$S_1P_2P_1$	—	$S_1S_2P_1$[a] $S_1(S_2)$[b]	$(S_1)_2$	irrotational
u_3, w_3	P_1S_1		$P_1P_2S_1$	—	$P_1S_2S_1$[a] S_2S_1[b]	$(S_1)_3$	equivoluminal
u_4, w_4	S_1S_1		$S_1P_2S_1$	$S_1P_1S_1$	$S_1S_2S_1$	—	equivoluminal

SH COMPONENT

v_1	S_1S_1				$S_1S_2S_1$	—	equivoluminal

TABLE II

DOMAINS OF EXISTENCE OF THE WAVES IN TABLE I
P, SV COMPONENTS

	Saddle Point	Roots $D=0$	$-k'$	$-k$	$-h'$	$-h$	
u_1, w_1	$z>0$	Determined by position of root	$\epsilon>i_1$	—	$\epsilon>i_3'$[a] $\epsilon>i_3$[b]	$\epsilon>i_5$	
u_2, w_2	$z>0$		$\epsilon_{21}>i_2$	—	$\epsilon_{21}>i_4$[a] $r/v'>$[b]	$r/v>$	$\dfrac{d}{v\cos\epsilon_{21}}+\dfrac{z}{v\cos\epsilon_{21}'}$
u_3, w_3	$z>0$		$\epsilon_{12}>i_1$	—	$\epsilon_{12}>i_3'$[a] $r/v'>$[b]	$r/v>$	$\dfrac{d}{V\cos\epsilon_{12}}+\dfrac{z}{v\cos\epsilon_{12}'}$
u_4, w_4	$z>0$		$\epsilon>i_2$	$\epsilon>i_5$	$\epsilon>i_4$	—	

SH COMPONENT

v_1	$z>0$				$\epsilon>i_4'$	—	

[a] $v'>V$, i.e. $k>h'$.
[b] $v'<V$, i.e. $k<h'$.

TABLE III
WAVE SYSTEMS IN THE LOWER MEDIUM
P, SV COMPONENTS

	First Order Waves		Second Order Waves				
	Saddle Point	Roots $D=0$	$-k'$	$-k$	$-h'$	$-h$	Wave Type
u_1', w_1'	$P_1\underline{P}_2$	Stoneley and pseudo-Rayleigh Waves	—	$(P_1)_1$	$P_1(\underline{S}_2)^a$ $(\underline{S}_2)_1^b$	$(\underline{S}_1)_1$	irrotational
u_2', w_2'	$S_1\underline{P}_2$		—	$S_1(\underline{P}_1)_2$	$S_1(S_2)$	$(\underline{S}_1)_2$	irrotational
u_3', w_3'	$P_1\underline{S}_2$		$P_1P_2\underline{S}_2$	$(\underline{P}_1)_3^a$	$(S_2)_3^b$	$(S_1)_3$	equivoluminal
u_4', w_4'	$S_1\underline{S}_2$		$S_1P_2\underline{S}_2$	$S_1(\underline{P}_1)_4^a$	$S_1P_1\underline{S}_2^b$	$(S_2)_4$	equivoluminal

SH COMPONENT

v'	$S_1\underline{S}_2$				—	$(\underline{S}_1)$	equivoluminal

TABLE IV
DOMAINS OF EXISTENCE OF THE WAVES IN TABLE III
P, SV COMPONENTS

	Saddle Point	Roots $D=0$	$-k'$	$-k$	$-h'$	$-h$	
u_1', w_1'	$z<0$	Determined by position of root	—	$r/V>$	$r/v'>$	$r/v>$	$\frac{d}{V\cos\eta_{11}} - \frac{z}{V'\cos\eta_{11}'}$
u_2', w_2'	$z<0$		—	$r/V>$	$r/v'>$	$r/v>$	$\frac{d}{v\cos\eta_{21}} - \frac{z}{V'\cos\eta_{21}'}$
u_3', w_3'	$z<0$		$\eta_{12}>i_1$	$r/V>^a$	$r/v'>^b$	$r/v>$	$\frac{d}{V\cos\eta_{12}} - \frac{z}{v'\cos\eta_{12}'}$
u_4', w_4'	$z<0$		$\eta_{22}>i_2$	$r/V>^a$	$\eta_{22}>i_5^b$	$r/v>$	$\frac{d}{v\cos\eta_{22}} - \frac{z}{v'\cos\eta_{22}'}$

SH COMPONENT

v'	$z<0$				—	$r/v>$	$\frac{d}{v\cos\eta_{22}} - \frac{z}{v'\cos\eta_{22}'}$

[a] $v'>V$, i.e. $k>h'$. [b] $v'<V$, i.e. $k<h'$.

boundary (Stoneley and pseudo-Rayleigh) waves, there exist *twelve* second-order waves in each medium, of which *seven* (when $V>v'$) or *nine* (when $V<v'$) are head waves in the upper medium, and *two* (when $V>v'$) or *three* (when $V<v'$) are head waves in the lower medium.

Tables II and IV list the domains of existence of the corresponding waves in Tables I and III respectively. The angles ϵ, ϵ_{12}, ϵ_{21}, ϵ_{12}', ϵ_{21}' of Table II are defined for each point (r, z) by the following sets of equations:

$$r = (z + d) \tan \epsilon$$

$$r = z \tan \epsilon_{12}' + d \tan \epsilon_{12}; \qquad V/\sin \epsilon_{12} = v/\sin \epsilon_{12}'$$

$$r = z \tan \epsilon_{21}' + d \tan \epsilon_{21}; \qquad v/\sin \epsilon_{21} = V/\sin \epsilon_{21}'.$$

Referring to Figure 6, it is clear that ϵ, ϵ_{12}, ϵ_{21} are angles of incidence, while

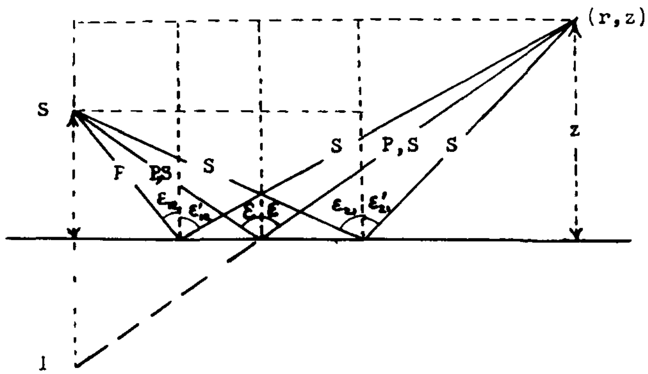

FIG. 6. Angles of incidence ϵ_{12} etc., and of reflection ϵ_{12}'.

$\epsilon' = \epsilon$, ϵ_{12}', ϵ_{21}' are the corresponding angles of reflection of the four reflected waves that reach the point (r, z) from the source.

Similarly, the angles η_{11} etc., η_{11}' etc. of Table IV are defined for each point (r, z) by the following sets of equations:

$$r = -z \tan \eta_{11}' + d \tan \eta_{11}; \qquad V/\sin \eta_{11} = V'/\sin \eta_{11}'$$

$$r = -z \tan \eta_{22}' + d \tan \eta_{22}; \qquad v/\sin \eta_{22} = v'/\sin \eta_{22}'$$

$$r = -z \tan \eta_{12}' + d \tan \eta_{12}; \qquad V/\sin \eta_{12} = v'/\sin \eta_{12}'$$

$$r = -z \tan \eta_{21}' + d \tan \eta_{21}; \qquad v/\sin \eta_{21} = V'/\sin \eta_{21}'.$$

Referring to Figure 7, it is clear that the angles η_{11} etc. (unprimed) are angles of incidence, and η_{11}' etc. (primed) are angles of refraction of the various refracted waves that reach the point (r, z) from the source.

The critical (or pseudo-critical) angles i_1, i_2 etc. are defined as follows:

$$i_1 = \arcsin V/V', \quad i_2 = v/V', \quad i_3 = \arcsin v'/V \text{ (when } V > v'),$$
$$i_3' = \arcsin V/v' \text{ (when } V < v'), \quad i_4 = \arcsin v/v',$$
$$i_5 = \arcsin v/V, \quad i_6 = \arcsin v'/V'.$$

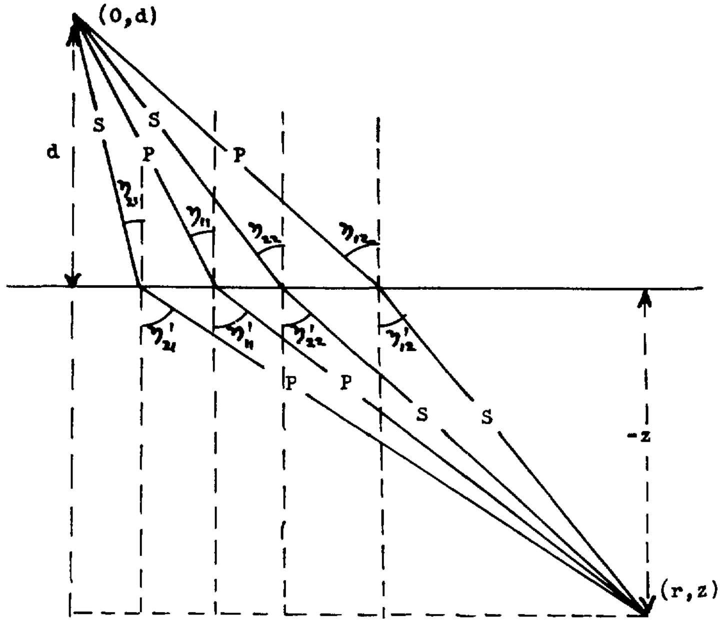

FIG. 7. Angles of incidence η_{12} etc., and of refraction η_{12}'.

STUDY OF A PARTICULAR HEAD-WAVE SYSTEM

In the preceding section, the complete set of first and second-order waves generated by the impact of a disturbance on a plane interface between two media, was obtained, and listed in two tables, with the domains of existence of these waves listed in two further tables. A careful scrutiny of the second-order waves, among which the various head waves are to be considered, shows that these are grouped together in systems which bear some relation to the critical angles of incidence of the impinging disturbance. Thus the head waves $P_1P_2P_1$ and $P_1P_2S_1$ in the upper medium, and $P_1P_2S_2$ in the lower medium all involve the amplitude of the incident critical P-ray, all seem to start on the boundary at the point $r=d \tan i_1$,[8] and travel for a certain distance with velocity V' along the boundary *with a harmony of phase* before branching out into their respective media as waves of head-wave type (see Fig. 8).

The analytical expressions for the three head waves just mentioned are

[8] It will be remembered that $i_1(=\arcsin V/V')$ is the critical angle of incidence of the refracted P-wave.

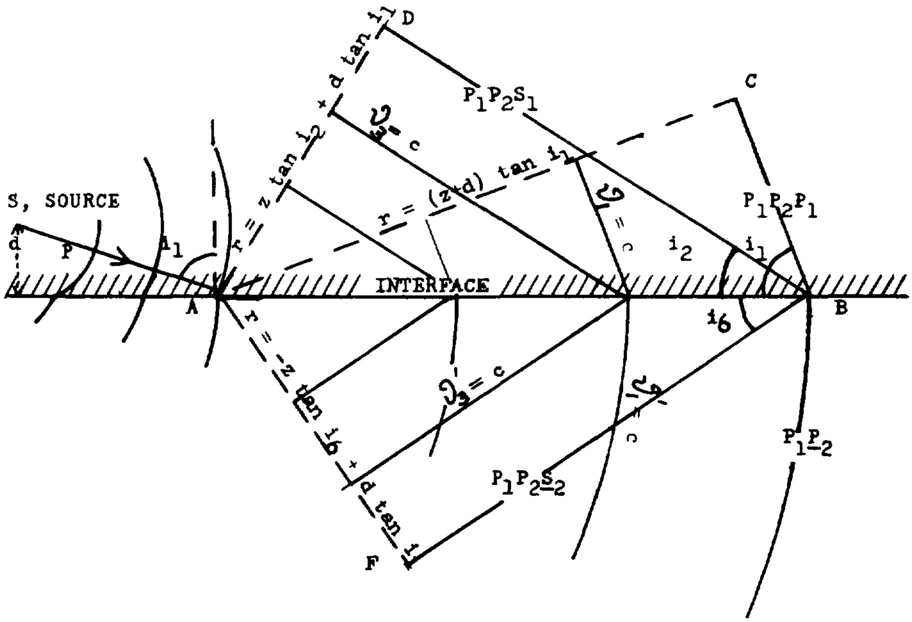

FIG. 8. Wave front diagram of the head wave system $P_1P_2P_1$, $P_1P_2S_1$, $P_1P_2\underline{S}_2$ associated with the refracted P-wave.

grouped together below. Those for $P_1P_2P_1$ were given in the preceding section. The others can be derived by an application of the method described in that section to the integrals (u_3, w_3) and (u_3', w_3').[9]

For $P_1P_2P_1$:

$$\begin{bmatrix} u_1(k') \\ w_1(k') \end{bmatrix} = \left[\frac{XF_1(i_1)p(\theta_1)}{r^{1/2}L_1^{3/2}}\right]\begin{bmatrix} \sin\ i_1 \\ \cos\ i_1 \end{bmatrix} \tag{26}$$

where

$$L_1 = r - (z + d)\tan i_1; \qquad \theta_1 = t - [(z + d)\cos i_1]/V - r/V';$$
$$X = [i\sigma(P_1Q - Q_1P)/Q^2 \cos i_1]_{-k'} \quad \text{and} \quad P_1,\ Q_1 \text{ etc.}$$

are algebraic expressions defined in equations (6) to (14).

For $P_1P_2S_1$:

$$\begin{bmatrix} u_3(k') \\ w_3(k') \end{bmatrix} = \left[\frac{YF_1(i_1)p(\theta_3)}{r^{1/2}L_3^{3/2}}\right]\cdot\begin{bmatrix} -\cos i_2 \\ \sin i_2 \end{bmatrix} \tag{27}$$

where

$$L_3 = r - z\tan i_2 - d\tan i_1;$$
$$\theta_3 = t - [z\tan i_2]/v - [d\cos i_1]/V - r/V';$$
$$Y = [\sigma h\tan i_1(P_3Q - Q_3P)/Q^2]_{-k'} \quad \text{and} \quad i_2 = \arcsin v/V'.$$

[9] *Cf.* equations (17) and (22).

For $P_1P_2S_2$:

$$\begin{bmatrix} u_3'(k') \\ w_3'(k') \end{bmatrix} = \begin{bmatrix} \dfrac{ZF_1(i_1)p(\theta_3')}{r^{1/2}L_3'^{3/2}} \end{bmatrix} \begin{bmatrix} \cos\ i_6 \\ \sin\ i_6 \end{bmatrix} \tag{28}$$

where

$$L_3' = r + z \tan i_6 - d \tan i_1;$$

$$\theta_3' = t + [z \cos i_6]/v' - [d \cos i_1]/V - r/V';$$

$$Z = [\sigma h' \tan i_1(P_3'Q - Q_3'P)/Q^2]_{-k'} \quad \text{and} \quad i_6 = \arcsin v'/V'.$$

It is of some interest to examine the relationship between these three head waves and the refracted P-wave, in view of the variety of opinions that exist as to the manner of generation of the head waves and the source of their energy. On this point, there are two schools of thought. One school (*cf.* Jeffreys, 1926; Muskat, 1933; Joos and Teltow, 1939; Scholte, 1946, 1947) regards the head waves as arising directly from the refracted wave by a process of "diffraction" at the boundary. The strongest objection to this view arises from the relatively large energy associated with $P_1P_2P_1$, a fact apparently repugnant to the physical notion of diffraction. The other school (*cf.* Macelwane, 1947; von Schmidt, 1936) regards the head waves as generated by a boundary wave of considerable energy in the lower medium, this wave itself owing its origin to the refracted P-wave at grazing incidence. This theory accounts quite satisfactorily for the observed strength of $P_1P_2P_1$. It introduces, however, a number of new and as yet unanswered questions regarding the nature and manner of origin of the boundary wave. A third opinion already disproved by the work of Jeffreys (1926), Muskat (1933), and Joos and Teltow (1939) was that the head wave was nothing other than a refracted P-wave deflected back in the direction from which it came by reason of a positive velocity gradient in the lower medium. While some of the energy in the head wave may, in fact, be obtained in this way, it has been shown that a positive velocity gradient is *not* a necessary condition for its existence.

In order to investigate the precise manner in which the head wave system comprising $P_1P_2P_1$, $P_1P_2S_1$ and $P_1P_2\underline{S}$ depends upon the refracted P-wave, it is necessary, first of all, to examine the behavior of that wave in the vicinity of the boundary. The integral yielding the amplitude of the refracted P-wave is[10]

$$u_1' = -\int_C \frac{\sigma f_0 D_1'}{D} H_1^{(1)}(\sigma r) e^{\alpha' z - \alpha d} d\sigma$$

$$w_1' = \int_C \frac{\alpha' f_0 D_1'}{D} H_0^{(1)}(\sigma r) e^{\alpha' z - \alpha d} d\sigma$$

where $\alpha = (\sigma^2 - k^2)^{1/2}$, $\alpha' = (\sigma^2 - k'^2)^{1/2}$, d is the distance of the source above the interface, and the functions D_1', D have been defined in the first section. Using the

[10] Equation (20).

method of steepest descent and carrying the asymptotic expansion to its second term, the amplitude of P_1P_2 is found to be,

$$u_1' \simeq \left[\frac{\tan \eta_{11}}{V \sin \eta_{11}'}\right]\left[\frac{\sin \eta_{11}}{rU_1}\right]^{1/2} B(\eta_{11})F_1(\eta_{11}) \left[\frac{d}{dt}\, p(\theta')\right] \sin \eta_{11}'$$

$$+ \left[\frac{\sin \eta_{11}}{rU_1}\right]^{1/2} \left[kE_1 \left\{\frac{D_1'f_0}{D}\right\} + kE_2 \frac{d}{d\sigma}\left\{\frac{D_1'f_0}{D}\right\}\right.$$

$$\left. + kE_3 \frac{d^2}{d\sigma^2}\left\{\frac{D_1'f_0}{D}\right\}\right]_{\sigma_0} p_1(\theta') \qquad (29a)$$

$$w_1' \simeq - \left[\frac{\tan \eta_{11}}{V \sin \eta_{11}'}\right]\left[\frac{\sin \eta_{11}}{rU_1}\right]^{1/2} B(\eta_{11})F_1(\eta_{11})\left[\frac{d}{dt}\, p(\theta)\right] \cos \eta_{11}'$$

$$- i\left[\frac{\sin \eta_{11}}{rU_1}\right]^{1/2} \left[kE_1\left\{\frac{\alpha' D_1' f_0}{\sigma D}\right\} + kE_2 \frac{d}{d\sigma}\left\{\frac{\alpha' D_1' f_0}{\sigma D}\right\}\right.$$

$$\left. + kE_3 \frac{d^2}{d\sigma^2}\left\{\frac{\alpha' D_1' f_0}{\sigma D}\right\}\right]_{\sigma_0} p_1(\theta') \qquad (29b);$$

where

$$\theta' = t + z/V' \cos \eta_{11}' - d/V \cos \eta_{11},$$

$\sigma_0' = -k' \sin \eta_{11}' = -k \sin \eta_{11}$, with $\alpha_0' = ik' \cos \eta_{11}'$, and $\alpha_0 = ik \cos \eta_{11}$;

$$4ikU_1^4E_1 \sin^2 \eta_{11} = 21U_2 \sin^2 \eta_{11} + 3U_3U_1 \sin^2 \eta_{11} + 6U_2U_1^2 \sin \eta_{11} + U_1^3$$

$$iU_1^2E_2 = -3U_2 + U_1 \sin \eta_{11}$$

$$U_1E_3 = ik$$

with

$$U_1 = -kz/k' \cos^3 \eta_{11}' + d/\cos^3 \eta_{11}$$

$$U_2 = -\frac{k^2 z \sin \eta_{11}'}{k'^2 \cos^5 \eta_{11}'} + \frac{d \sin \eta_{11}}{\cos^5 \eta_{11}}$$

$$U_3 = -\frac{k^3 z(\cos^2 \eta_{11}' + 5 \sin^2 \eta_{11}')}{k'^3 \cos^7 \eta_{11}'} + \frac{d(\cos^2 \eta_{11} + 5 \sin^2 \eta_{11})}{\cos^7 \eta_{11}}$$

and $B(\eta_{11}) = [D_1'/D]_{\sigma_0'}$-refraction coefficient of P incident at angle η_{11} and transmitted as a P-wave, i.e., ratio of amplitudes of associated Knott functions, using the definition given by Slichter and Gabriel, (1933).[11]

To find the displacement produced by the refracted wave in the neighborhood of the boundary, beyond the cone of critical incidence, it is necessary to let $z \to 0$, $\eta_{11}' \to \pi/2$, $\eta_{11} \to i_1 = \arcsin V/V'$, and $-z \tan \eta_{11}' \to [r - z \tan i_1] = L$. Thus

[11] $B(\eta_{11})$ of the present notation corresponds to the A' used by Slichter and Gabriel; η_{11} is the angle of incidence.

$$U_1^{-1} \rightarrow [k' \cos^2 \eta_{11}']/kL \rightarrow 0.$$

The first term, consequently, in the expressions for u_1' and w_1' vanishes on the boundary and the character of the wave there is determined by the terms of the second-order. These reduce to

$$u_1' \simeq \frac{W_1F_1(i_1)p(\theta)}{r^{1/2}L^{3/2}}$$

$$w_1' \simeq \frac{W_2F_1(i_1)p(\theta)}{r^{1/2}L^{3/2}}$$

where

$$W_1 = ik' \tan i_1[PD_1'/Q^2]_{-k'}$$
$$W_2 = \quad \tan i_1[D_1'/Q]_{-k'}.$$

It should be noted that the vertical component of displacement is not zero. Thus, the particle displacement is not perfectly longitudinal to the path determined by geometrical optics, nor is the flow of energy along the geometrical ray.

It might be well to interrupt the discussion here to say a few words on the nature of first and second order waves. A localized source of seismic energy radiates a disturbance which is propagated outwards through a family of closed wave fronts that approach more and more closely to the spherical type with increasing distance from the source. The particle displacement in such a wave cannot be represented by a single term, as is the case with a plane wave, but it may be expanded in an asymptotic series in inverse powers of the distance from the source. When the first and predominant term is of the order of R^{-1} at infinity, the wave is called a first order wave.[12] It is clear from equations (29a) and (29b) that the refracted P-wave, $P_1\underline{P_2}$, is of this type. When the predominant term is of the order of R^{-2} at infinity the wave is called a second order wave. It will be seen without much difficulty that the waves specified by equations (26), (27), and (28) are of the latter type.

It can happen that the first order term of a wave like $P_1\underline{P_2}$ vanishes at certain points, or within a certain domain. In this case, the nature of the wave is governed by the second order term, and this may have properties different from those deducible from the first term alone. For example, we have seen that on the boundary, outside the cone of critical incidence, the particle displacement and energy flow in the refracted P-wave is governed by the second-order term, and that it is not directed *longitudinally* along the geometrical ray, which here lies parallel to the interface. The wave however, is still a dilatational wave, for its curl is zero.

[12] Considering only the term of the first order, such waves have properties closely resembling those of plane waves, obeying Snell's Law, diffusing energy normal to the wave front, and propagating energy to infinity.

Returning to our discussion of the relationship between the refracted P-wave and the three head waves, it can be seen that on the boundary ($z=0$), outside the cone of critical incidence (i.e. $r>d \tan i_1$),

$$\theta_1 = \theta_3 = \theta_3' = \theta' = t - [d \cos i_1]/V - r/V'.$$

The four waves $P_1\underline{P}_2$, $P_1P_2P_1$, $P_1P_2S_1$, and $P_1P_2\underline{S}_2$ are, consequently, in phase with one another on the boundary. A comparison of the displacement vectors shows moreover, that the vector sum of the displacements of the two waves in the upper medium, *viz.* $P_1P_2P_1$ and $P_1P_2S_1$, equals the vector sum of the displacements of the two waves in the lower medium, *viz.* $P_1\underline{P}_2$ and $P_1P_2S_2$, where the four waves overlap on the boundary, i.e.

$$u_1(k') + u_3(k') = u_1' + u_3'(k')$$
$$w_1(k') + w_3(k') = w_1' + w_3'(k')$$

when $z=0$. Utilized in these results are the four identities:

$$\alpha P_1 + \sigma^2 P_3 - \sigma^2 P_3' + D_1' \equiv \alpha P$$
$$\alpha Q_1 + \sigma^2 Q_3 - \sigma^2 Q_3' \equiv \alpha Q$$
$$P_1 + \beta P_3 + \beta' P_3' \equiv - P$$
$$Q_1 + \beta Q_3 + \beta' Q_3' - D_1' \equiv - Q.$$

Similarly, the stresses exerted on the underside of the boundary by $P_1\underline{P}_2$ and $P_1P_2\underline{S}_2$ are continuous with those exerted by $P_1P_2P_1$ and $P_1P_2S_1$ on the upper side. Thus, from a physical point of view, the system of four waves, satisfying conditions of continuity of stress and displacement across the interface, represents a complete and independent dynamic system. No boundary wave appears, and none is required to satisfy the conditions of a real dynamic system propagated in the two media and linked dynamically across the boundary.

Origin of head wave energy.—Consider the flow of energy into a small box $AA'BB'$ (see Figure 9) set astride the boundary and bounded above and below by elements of area parallel to the boundary, and such that the dimensions of

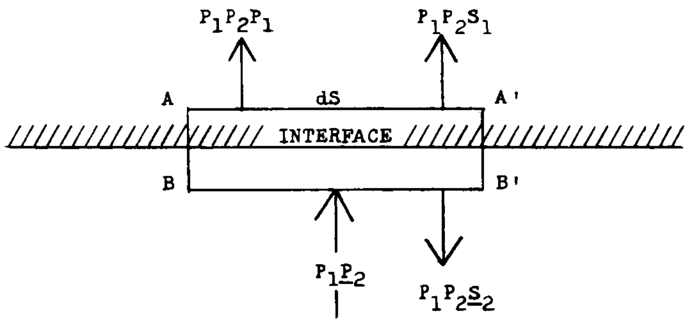

FIG. 9. Flow of energy into and out of box on boundary.

the sides AB and $A'B'$ are arbitrarily small. As no energy is created or destroyed within the box, the total energy leaving the box must equal the total energy entering it. It is found that the only energy entering the box is due to the vertical component of energy flow in P_1P_2, and has the value (integrated over the duration of the disturbance),

$$- \frac{\mu' F_1(i_1)^2}{rL^3} \left[\frac{ih'^2 PD_1'^2}{Q^3} \right]_{-k'} \frac{\tan^2 i_1}{k'V'} \int_{-\infty}^{+\infty} \left[\frac{d}{dt} p(t) \right]^2 dtdS. \tag{30}$$

The total energy leaving the box in the waves $P_1P_2P_1$, $P_1P_2S_1$ and $P_1P_2S_2$ is

$$- \frac{F(i_1)^2}{rL^3} [\{ i\mu L^2\alpha(P_1Q - Q_1P)^2 - i\mu h^2\sigma^2\beta(P_3Q - Q_3P)^2$$

$$- i\mu' h'^2\sigma^2\beta'(P_3'Q - Q_3'P)^2 \}/Q^4]_{-k'} \frac{\tan^2 i_1}{k'V'} \int_{-\infty}^{+\infty} \left[\frac{d}{dt} p(t) \right]^2 dtdS. \tag{31}$$

With the help of the following identity,

$$\mu h^2\alpha(P_1Q - Q_1P)^2 - \mu h^2\sigma^2\beta(P_3Q - Q_3P)^2$$
$$- \mu' h'^2\sigma^2\beta'(P_3'Q - Q_3'P)^2 \equiv \mu' h'^2 PQD_1'^2,$$

it is seen immediately that expression (31) is identical with (30), the energy entering the box through P_1P_2. We conclude therefore, that the energy of the three head waves $P_1P_2P_1$, $P_1P_2S_1$, and $P_1P_2S_2$ is derived solely and entirely from the refracted P-wave, not by virtue of the term which gives this wave its specific character at points located off the boundary (*viz.* the first order term), but by virtue of the term of the second order, a term, incidentally, which arises only when the wave fronts are curved.

We might enquire whether this process can legitimately be called *diffraction.* In the first place, the similarity between the mathematical formulation of the problem of light diffraction by a straight edge and its solution by Sommerfeld and others,[13] and the theory of head waves given here, is very striking and strongly urges the idea that the physical processes involved are analogous. There is one difference, however, that on account of the peculiar expression for the amplitudes of the head waves involving a factor in the denominator that can become very small, and because of the smaller frequencies involved, the head wave amplitude is not necessarily a small quantity compared with the amplitude of the incident radiation, as is the case with the diffracted light ray. Whether or not this destroys the argument for the close analogy between the two processes is largely a matter of personal opinion. We made use of the term head wave in this series of papers, partly to avoid having to make a decision on this matter, but for the most part, because this term seems better suited than any other to describe this phenomenon clearly and concretely.

[13] For an account of the problem and its solution, see Born, (1933, pp. 209–218).

It remains to compute the amplitudes from formulas (26), (27) and (28), and to compare these with the observed values. Table V lists the computed values for two of the cases considered by Slichter and Gabriel (1933) in their work on reflection coefficients. The task of comparing these with experimental values has not yet been undertaken.

TABLE V

AMPLITUDE COEFFICIENTS OF $P_1P_2P_1$, $P_1P_2S_1$, $P_1P_2S_2$ AND P_1P_2

	X	Y	Z	W_1	W_2
$\rho/\rho'=0.965$, $V/V'=v/v'=0.935$, Poisson's ratio$=0.25$, $i_1=69°$	17.0	1.3	0.0	17.3	7.2
$\rho/\rho'=0.8$, $V/V'=v/v'=0.75$, Poisson's ratio$=0.25$, $i_1=49°$	6.1	1.9	0.5	5.9	2.8

ACKNOWLEDGEMENT

In conclusion, I should like to thank the Reverend James B. Macelwane, S.J. of Saint Louis University, who suggested and directed this work, and who helped by his kind advice and encouragement to bring it to a successful conclusion.

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